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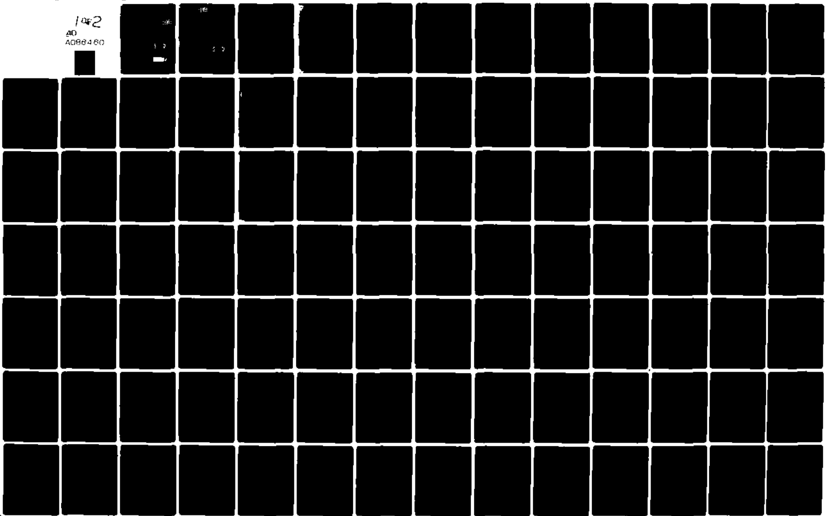

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APPENDIX A

DOA ACCURACY ANALYSIS

One technique for estimation of direction-of-arrival (DOA) of acoustic or electromagnetic energy is the use of sensors or antennas whose sensitivity is a function of sine and cosine of DOA of the incident energy. The signals received with an omnidirectional sensor and two such sinusoidal sensors (for a tone in noise) are given below:

$$e_o(t) = \sqrt{\frac{2}{\pi}}\, VB(t)\cos\left[\omega t + \emptyset(t)\right] + n_o(t) \qquad \text{(A-1)}$$

$$e_s(t) = \sqrt{\frac{2}{\pi}}\, VB(t)\sin\theta\, \cos\left[\omega t + \emptyset(t)\right] + n_s(t) \qquad \text{(A-2)}$$

$$e_c(t) = \sqrt{\frac{2}{\pi}}\, VB(t)\cos\theta\, \cos\left[\omega t + \emptyset(t)\right] + n_c(t) \qquad \text{(A-3)}$$

where $n_o(t)$, $n_s(t)$, and $n_c(t)$ are assumed zero mean Gaussian noise terms, $\emptyset(t)$ is a uniformly distributed (0 to 2π) random variable, and B(t) is a normalized Rayleigh distributed fading factor. The expected value of B(t) is $\sqrt{\frac{\pi}{2}}$ and the expected value of the noise-free envelope of the output of the omni channel is V. The processing assumed for extraction of the bearing information, θ, involves the cross-correlation of the omni channel with both the sine and cosine channels which removes terms at ω and 2ω and yields:

$$e_{os}(t) = \frac{1}{\pi} V^2 B^2(t) \sin\theta + \frac{1}{\sqrt{2\pi}} VB(t) n_{si}(t) + \frac{1}{\sqrt{2\pi}} VB(t) n_{oi}(t) \sin\theta$$

$$+ \frac{1}{2} n_{oi}(t)\, n_{si}(t) + \frac{1}{2} n_{oq}(t)\, n_{sq}(t) \qquad \text{(A-4)}$$

$$e_{oc}(t) = \frac{1}{\pi} V^2 B^2(t) \cos\theta + \frac{1}{\sqrt{2\pi}} VB(t) n_{ci}(t) + \frac{1}{\sqrt{2\pi}} VB(t) n_{oi}(t) \cos\theta$$

$$+ \frac{1}{2} n_{oi}(t)\, n_{ci}(t) + \frac{1}{2}\, n_{oq}(t)\, n_{cq}(t) \qquad \text{(A-4)}$$

The noise terms follow from an expansion of the input noise into their in-phase and quadrature components, that is

$$n_o(t) = n_{oi}(t) \cos(\ t + \emptyset) + n_{oq}(t)\ \sin(\ t + \emptyset) \qquad \text{(A-6)}$$

$$n_s(t) = n_{si}(t) \cos(\ t + \emptyset) + n_{sq}(t)\ \sin(\ t + \emptyset) \qquad \text{(A-7)}$$

$$n_c(t) = n_{ci}(t) \cos(\ t + \emptyset) + n_{cq}(t)\ \sin(\ t + \emptyset) \qquad \text{(A-8)}$$

If we assume that the noise field is isotropic, it is easy to show that the terms $n_o(t)$, $n_s(t)$, and $n_c(t)$ are mutually uncorrelated as follows: The noise terms may be defined as the integrals of the noise field over the full 360^o of azimuth (we are assuming here that the noise field is adequately represented by one dimension):

$$n_o(t) = \int_0^{2\pi} \nu_o(\theta,t)\, d\theta \qquad \text{(A-9)}$$

The cross product $n_o(t)n_s(t)$ can then be written in terms of $\nu_o(\theta,t)$:

$$n_o(t)n_s(t) = \int_0^{2\pi}\int_0^{2\pi} \nu(\theta_1,t)\, \nu_o\,(\theta_2,t) \sin\theta_1\, d\theta_1\, d\theta_2 \qquad \text{(A-10)}$$

The expected value of this expression can be written as:

$$E\left[n_o(t)n_s(t)\right] = \int_0^{2\pi}\int_0^{2\pi} E\left[\nu_o(\theta_1,t)\nu_o(\theta_2,t)\right] \sin\theta_1 \, d\theta_1 \, d\theta_2 \qquad \text{(A-11)}$$

In general, noise terms originating from different azimuthal bearings will be uncorrelated (discounting, for example, focused back-scatter from a rough bottom). With the noise assumed stationary, the expected value can be written in terms of a delta function:

$$E\left[n_o(t)n_s(t)\right] = \int_0^{2\pi}\int_0^{2\pi} N(\theta_1)\delta(\theta_1 - \theta_2) \sin\theta_1 \, d\theta_1 \, d\theta_2$$

$$= \int_0^{2\pi} N(\theta) \sin\theta \, d\theta \qquad \text{(A-12)}$$

For the isotropic noise field assumed here, the azimuthal noise power density is independent of θ and can be brought outside the integral as a constant:

$$E\left[n_o(t)n_s(t)\right] = N\int_0^{2\pi} \sin\theta \, d\theta = 0$$

Similar arguments show that for an isotropic noise field the omni, sine, and cosine channel noise terms are all mutually uncorrelated (and, therefore, statistically independent under a Gaussian noise assumption).
With these results, the expected values of $e_{os}(t)$ and $e_{oc}(t)$ are readily determined:

$$E\left[e_{os}(t)\right] = (1/\pi)V^2 \sin\theta \, E\left[B^2(t)\right] \qquad \text{(A-13)}$$

$$E\left[e_{oc}(t)\right] = (1/\pi)V^2 \cos\theta \, E\left[B^2(t)\right] \qquad \text{(A-14)}$$

Under the Rayleigh fading assumption, the expected value of B^2 equals 2 and these relations can be rewritten:

$$E\left[e_{os}(t)\right] = (2/\pi)V^2 \sin\theta \tag{A-15}$$

$$E\left[e_{oc}(t)\right] = (2/\pi)V^2 \cos\theta \tag{A-16}$$

Also under the Rayleigh fading assumption, the expected value of B^4 equals 8 and the variance of these "signal" terms are:

$$\sigma_s^2 = E\left\{\left[(1/\pi)V^2B^2 \sin\theta\right]^2 - \left[(2/\pi)V^2 \sin\theta\right]^2\right\} \tag{A-17}$$

$$\sigma_s = (2/\pi)V^2 |\sin\theta| \tag{A-18}$$

The similar result for the cosine channel is:

$$\sigma_c = (2/\pi)V^2 |\cos\theta|$$

With these results, it is convenient to rewrite $e_{os}(t)$ and $e_{oc}(t)$ in terms of a random variable $\varepsilon(t)$ with zero mean and unit variance:

$$e_{os}(t) = (2/\pi)V^2 \sin\theta\,(1 + \epsilon(t)) + n_{os}(t) \tag{A-19}$$

$$e_{oc}(t) = (2/\pi)V^2 \cos\theta\,(1 + \epsilon(t)) + n_{oc}(t) \tag{A-20}$$

where $\epsilon(t) \equiv \frac{B^2(t)}{2} - 1$

$$n_{os}(t) = VB(t)\left[n_{si}(t) + n_{oi}(t)\sin\theta\right]/\sqrt{2\pi} + \left[n_{oi}(t)n_{si}(t) + n_{oq}(t)n_{cq}(t)\right]/2 \tag{A-21}$$

$$n_{oc}(t) = VB(t)\left[n_{ci}(t) + n_{oi}(t)\cos\theta\right]/\sqrt{2\pi} + \left[n_{oi}(t)n_{ci}(t) + n_{oq}(t)n_{cq}(t)\right]/2$$

(A-22)

The noise terms $n_{os}(t)$ and $n_{oc}(t)$ clearly have zero means, since n_o, and n_s are uncorrelated. As is well known, the quadrature components are also uncorrelated, if a symmetric noise spectrum is assumed. The variance can be determined by finding the expected value of their squares:

$$\begin{aligned}\sigma_{os}^2 &= E\left[(n_{os})^2\right] \\ &= V^2 E\left[B^2\right]\left(E\left[n_{si}^2\right] + E\left[n_{oi}^2\right]\sin^2\theta\right)\Big/2\pi \\ &\quad +\left(E\left[n_{oi}^2\right]E\left[n_{si}^2\right] + E\left[n_{oq}^2\right]E\left[n_{sq}^2\right]\right)\Big/4\end{aligned} \tag{A-23}$$

$$\begin{aligned}\sigma_{oc}^2 &= E\left[(n_{oc})^2\right] \\ &= V^2 E\left[B^2\right]\left(E\left[n_{ci}^2\right] + E\left[n_{oi}^2\right]\cos^2\theta\right)\Big/2\pi \\ &\quad +\left(E\left[n_{oi}^2\right]E\left[n_{ci}^2\right] + E\left[n_{oq}^2\right]E\left[n_{cq}^2\right]\right)\Big/4\end{aligned} \tag{A-24}$$

It can be shown that the variances of the in-phase and quadrature noise amplitude components equal the variance of the entire noise process, that is

$$E\left(n_o{}^2\right) = E\left[n_{oi}{}^2\right] = E\left[n_{oq}{}^2\right] = \sigma_o^2 \tag{A-25}$$

$$E\left(n_s{}^2\right) = E\left[n_{si}{}^2\right] = E\left[n_{sq}{}^2\right] = \sigma_s^2 \tag{A-26}$$

$$E\left(n_c{}^2\right) = E\left[n_{ci}{}^2\right] = E\left[n_{cq}{}^2\right] = \sigma_c^2 \tag{A-27}$$

These values can be determined by extending the results of Equation A-12. The variance σ_o^2 results by replacing $n_s(t)$ with $n_o(t)$. This eliminates the sine function in the integral and yields $\sigma_o^2 = 2\pi N$. Replacing $n_o(t)$ with $n_s(t)$ in Equation A-12 would yield σ_s^2. The integral would contain

a sine^2 function and yield $\sigma_s^2 = 2\pi N/2$. Similarly σ_c^2 can be shown equal to σ_s^2. Therefore, for an isotropic noise field:

$$\sigma_o^2 = \sigma^2 = 2\pi N \tag{A-27a}$$

$$\sigma_s^2 = \sigma^2/2 \tag{A-27b}$$

$$\sigma_c^2 = \sigma^2/2 \tag{A-27c}$$

These expressions can be used to evaluate Equations A-23 and A-24:

$$\sigma_{os}^2 = 2V^2\left[(\sigma^2/2) + \sigma^2 \sin^2\theta\right]\Big/2\pi + \left[(\sigma^2\sigma^2/2) + (\sigma^2\sigma^2/2)\right]\Big/4$$

$$= V^2\sigma^2(1+2\sin^2\theta)\Big/2\pi + \sigma^4\Big/4 \tag{A-28}$$

$$\sigma_{oc}^2 = V^2\sigma^2(1 + 2\cos^2\theta)\Big/2\pi + \sigma^4\Big/4 \tag{A-29}$$

The numbers s(M) and c(M) result from averaging M independent samples of the processes $e_{os}(t)$ and $e_{oc}(t)$, respectively:

$$S(M) = (2/\pi)V^2 \sin\theta(1 + \varepsilon_M) + n_{sM} \tag{A-30}$$

$$C(M) = (2/\pi)V^2 \sin\theta(1 + \varepsilon_M) + n_{cM} \tag{A-31}$$

where

$$E\left[\varepsilon_M\right] = E\left[n_{sM}\right] = E\left[n_{cM}\right] = 0 \tag{A-32a,b,c}$$

$$E\left[\varepsilon_M^2\right] = 1/M \tag{A-32d}$$

$$E\left[n_{sM}^2\right] = \sigma_{os}^2\Big/M \tag{A-32e}$$

$$E\left[n_{cM}^2\right] = \sigma_{oc}^2\Big/M \tag{A-32f}$$

To estimate the DOA, θ, the ratio S(M)/C(M) is formed.

$$S(M)/C(M) = \left[(2/\pi)V^2\sin\theta(1 + \varepsilon_M) + n_{sM}\right] \Big/ \left[(2/\pi)V^2\cos\theta(1 + \varepsilon_M) + n_{cM}\right] \tag{A-33}$$

In the noise-free case this expression reduces to sinθ/cosθ or tanθ which can be used directly to estimate signal bearing angle. In the presence of noise, the use of this tangent relationship will lead to errors in estimation of the bearing angle θ.

A number of authors have studied the errors associated with a bearing estimator such as:

$$\hat{\theta} = \tan^{-1}\left[S(M)/C(M)\right] \tag{A-34}$$

In particular, Richter (Ref. A-1) studied the following estimator:

$$\hat{\theta} = \tan^{-1}\left[\frac{A\sin\theta + n_y(M)}{A\cos\theta + n_x(M)}\right] \tag{A-35}$$

His results can be applied to the estimator of Equation A-34 using a redefinition of terms appropriate to our problem:

$$A = 2V/\pi \tag{A-36}$$

$$n_y(M) = n_{sM}\Big/\left(1 + \epsilon_M\right) \tag{A-37}$$

$$n_x(M) = n_{cM}\Big/\left(1 + \epsilon_M\right) \tag{A-38}$$

In writing Equations A-37 and A-38 we recognize the fact, whereas the terms $(1 + \epsilon_M)$ are associated with the signal components in forming S(M) and C(M), the effect of this term is equivalently an inverse scaling of the noise terms n_{sM} and n_{cM}.

In order to apply Richter's results, the terms $n_x(M)$ and $n_y(M)$ should be independent Gaussian variables We notice that, since n_{sM} and n_{cM} are uncorrelated, $n_x(M)$ and $n_y(M)$ are also uncorrelated. For large M, the terms n_{sM}, n_{cM}, and ϵ_M may also be assumed to be normally distributed. The ratios used to define $n_y(M)$ and $n_x(M)$, however, are not normally distributed. For large M, the variance of ϵ_M becomes quite small and in the limit, as $M \to \infty$, goes to zero. In the limit, therefore, both $n_y(M)$ and $n_x(M)$ approach a normal distribution. In the analysis that follows we will assume M is large enough for the discripancy to be negligible and will assume Richter's results apply exactly.

The pertinent portion of Richter's paper is presented in Appendix B. In order to use Richter's results, two terms, S^2 and a^2, must be defined:

$$S^2 = A^2/\left(\sigma_x^2 + \sigma_y^2\right) \tag{A-39}$$

and

$$a^2 = \sigma_y^2/\sigma_x^2 \tag{A-40}$$

where

$$\sigma_x^2 = E\left[n_x^2\right] = E\left[n_{cM}^2\right] E\left[1/\left(1 + \epsilon_M\right)^2\right] \tag{A-41}$$

$$\sigma_y^2 = E\left[n_y^2\right] = E\left[n_{sM}^2\right] E\left[1/\left(1 + \epsilon_M\right)^2\right] \tag{A-42}$$

For small ϵ_M (i.e. large M) the term $E\left[1/\left(1 + \epsilon_M\right)^2\right]$ can be estimated by expanding $1/\left(1 + \epsilon_M\right)^2$:

$$E\left[1/\left(1 + \epsilon_M\right)^2\right] = E\left[1 - 2\epsilon_M + 3\epsilon_M^2 + 0\left(\epsilon^3\right)\right]$$
$$\approx 1 + \frac{3}{M} \tag{A-43}$$

Substituting σ_x^2, σ_y^2 (using Equations A-28, A-29 and A-43) and K^2, into Equations A-39 and A-40 yields

$$S^2 = \left(\frac{M^2}{M + 3}\right)\left(\frac{2V^2}{\pi}\right)^2 \left(\frac{2V^2\sigma^2}{\pi} + \frac{\sigma 4}{2}\right) \tag{A-44}$$

$$a^2 = \frac{V^2\sigma^2(1 + 2\sin^2\theta)/2\pi + \sigma^4/4}{V^2\sigma^2(1 + 2\cos^2\theta)/2\pi + \sigma^4/4} \tag{A-45}$$

Using Eq. A-1, the signal-to-noise-ratio in the omni channel can be defined:

$$\gamma = (2V^2/\pi)/\sigma^2 \tag{A-46}$$

This expression can be used to simplify Equations A-44 and A-45:

$$S^2 = \left(\frac{M^2}{M + 3}\right)\gamma^2 \Big/ \left(\gamma + \frac{1}{2}\right) \tag{A-47}$$

$$a^2 = \left[\gamma(1 + 2\sin^2\theta) + 1\right] \Big/ \left[\gamma(1 + 2\cos^2\theta) + 1\right] \tag{A-48}$$

In general, M (the required number of samples averaged) is the parameter which determines system integration time. This parameter can be determined from Equation A-47:

$$M = \frac{r}{2}\left[1 + \sqrt{1 + \frac{12}{r}}\right] \tag{A-49}$$

$$M \approx \frac{r}{2}\left[1 + \left(1 + \frac{6}{r}\right)\right] \qquad \text{for } \frac{8}{r} < \frac{1}{3}$$

$$M \approx r + 3 \tag{A-50}$$

where $\qquad r \equiv S^2(1 + 2\gamma)/(2\gamma^2)$

For large M, $M \approx r$ and the condition necessary for the square root approximation becomes

$$\frac{12}{M} < \frac{1}{3}$$

or $$M > 36$$

For M = 36, Equation A-49 is within 10% of the exact expression Equation A-48, with the error decreasing to 0 for large M. Also for $M \geq 36$, Equation A-43 is a very tight estimate of the expected value of $1/\left(1 + \epsilon_M\right)^2$.

Equation A-49 and the curves of Appendix B were used to generate the curves of Figure A-1, which give rms bearing error as a function of the expected input omni-directional SNR and the number of samples M for both the maximum error bearing directions (45^o, 135^o, 225^o, 315^o) and the minimum error bearing direction (0^o, 90^o, 180^o, 270^o). In addition, since Richter's curves were not readable for the case of very small error, a small noise approximation was derived using Equations A-33, A-34, and A-35:

$$M = \left[\frac{3 - \cos 4\theta}{8\gamma} + \frac{1}{(4\gamma)^2}\right] / \sigma_\theta^2 \tag{A-51}$$

where σ_θ is the desired rms bearing accuracy. This expression was used to generate the 1^o rms curves of Figure A-1. As an example of the derivation of these curves and the use of the curves of Appendix B, the following set of parameters are used to determine M:

$$\sigma_\theta = 10^o \text{ rms}, \ (\sigma_\theta^2 = 0.03 \text{ rad}^2)$$
$$\gamma = -5 \text{ dB}, \ (\gamma = 0.316)$$
$$\sigma_\theta^2 = 45^o, \ 0^o$$

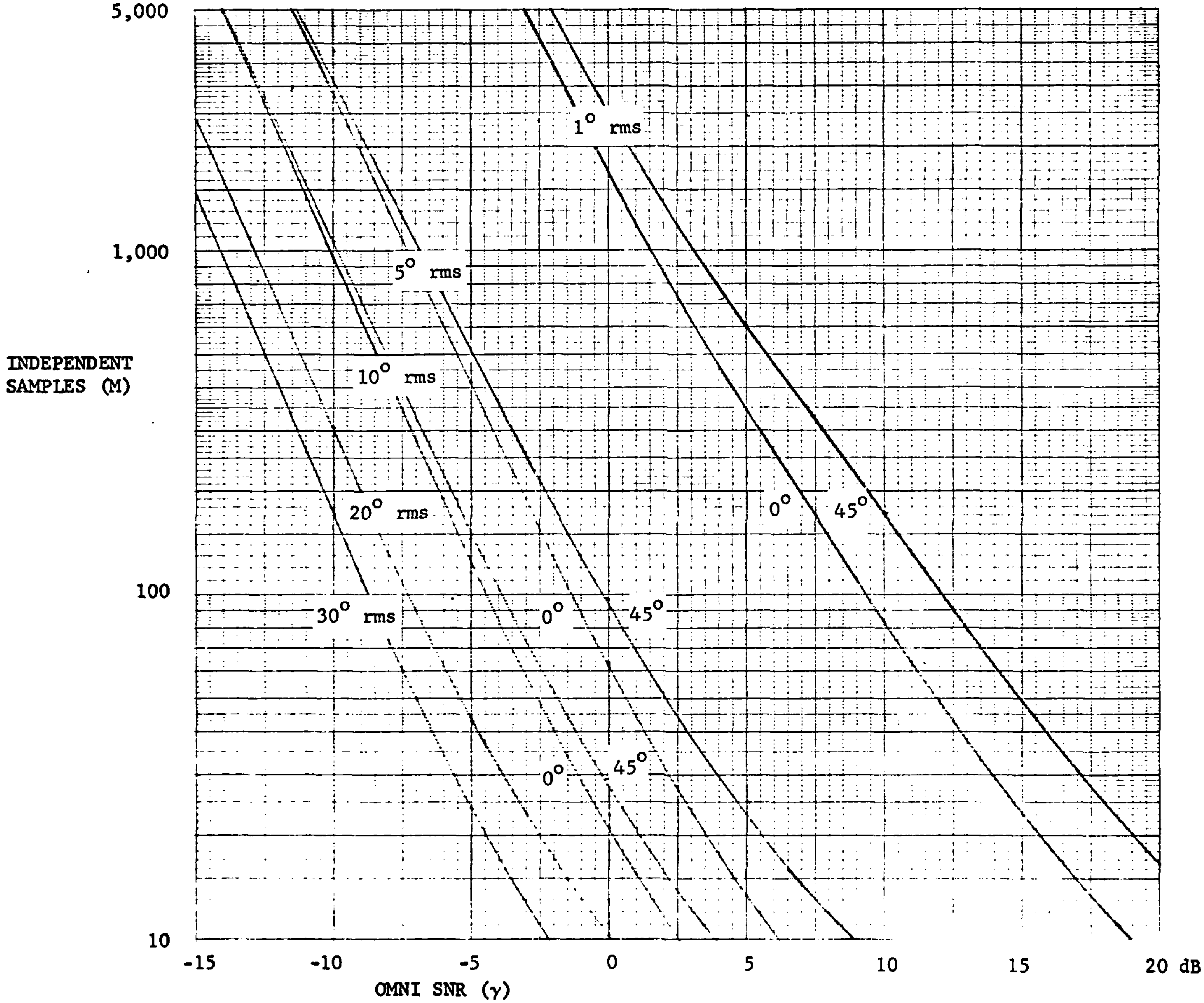

FIGURE A-1 (U) DOA ACCURACY WITH SIN, COS, AND OMNI SENSORS (U)

The first parameter to be calculated is a. Using Equation A-45, it is easily shown that at 45°, $a = 1$ and at 0°, $a = 0.82$. With these values of σ_θ^2 and a, Figures 3-2a (page B-8) and 4-2a (page B-9) can be used to determine S^2. The resulting values are:

$$S^2 = 12.5 \text{ dB } (17.78), \ \theta = 45^\circ$$

$$S^2 = 11.5 \text{ dB } (14.12), \ \theta = 0^\circ$$

Using these values in Equation A-46 yields the desired values of M:

$$M = 147 \text{ samples}, \ \theta = 45^\circ$$

$$M = 117 \text{ samples}, \ \theta = 0^\circ$$

Finally, Figure 4-3 (page B-13) and Table 4-1 (page B-6) can be used to estimate the bias error resulting from S^2 at $\theta = 45^\circ$ and 0°. These figures show the resulting bias to be negligible at 0° and 45° $(\pm n\pi/2)$.

For this same set of parameters, the approximate expression, Equation A-47, yields:

$$M = 136 \text{ samples}, \ \theta = 45^\circ$$

$$M = 110 \text{ samples}, \ \theta = 0^\circ$$

The agreement between the numerically derived results and the approximate results at 10° rms (better than 8%) suggests that the approximate expression is very tight at 1° rms and for many cases of practical interest may be used without significant error up to 10° rms.

APPENDIX B

AVERAGING TO REDUCE VARIANCE IN BEARING ESTIMATES

This appendix is a reproduction of part of Section 4 and Figure 3-22 of a 1969 Magnavox Technical Report (MX-TR-6-7-200169) of the same title which was prepared by R. J. Richter and G. E. Gaerte. The report derives accuracy expressions that result from the use of x(t) and y(t) to estimate bearing. These are defined by

$$x(t) \equiv A\cos\theta + n_x(t)$$

$$y(t) \equiv A\sin\theta + n_y(t)$$

where θ is the true signal bearing. The actual bearing estimator analyzed is

$$\phi = \tan^{-1}(y/x)$$

In the analysis, the noise terms are assumed to be zero mean, Gaussian, ergodic processes with unequal variances σ_x^2 and σ_y^2. In the sections reproduced here, three equations from other sections are referenced, 1-30, 2-21, and 2-22. Equation 1-30 is identically Equation 4-1 of the enclosed section. Equations 2-21 and 2-22 define the mean and variance of ϕ and are repeated here:

$$E[\phi] = \int_{\theta-\pi}^{\theta+\pi} \phi P_\phi(\phi)d\phi \qquad (2\text{-}21)$$

$$\mathrm{var}[\phi] = \int_{\theta-\pi}^{\theta+\pi} \phi^2 P_\phi(\phi)d\phi - E^2[\phi] \qquad (2\text{-}22)$$

where P_ϕ is the probability density function of the estimator ϕ.

There are also two typographical errors in the material reproduced. The exponent of the second exponential expression in Equation 4-1 should be negative (i.e., "$-S^2(1 + a^2)\ldots$") and the exponential term in Equation 4-16 should be scaled by a factor of 1/2.

4.0 CASE OF UNEQUAL VARIANCES, ZERO CORRELATION

Now consider the case when X and Y are uncorrelated with unequal variances. Again we will first consider the estimator $\hat{\theta}$ for n = 1. The density function for $\hat{\theta}$ is given by equation (I-30)

$$P_\phi(\phi) = \frac{a}{2\pi(a^2 \cos^2\phi + \sin^2\phi)} \exp\left\{\frac{-S^2(1 + a^2)(a^2 \cos^2\theta + \sin^2\theta)}{2a^2}\right\}$$

$$+ \frac{S\sqrt{1 + a^2}\,(a^2 \cos\phi \cos\theta + \sin\phi \sin\theta)}{2\sqrt{2\pi}\,(a^2 \cos^2\phi + \sin^2\phi)^{3/2}}$$

$$\times \exp\left\{\frac{S^2(1 + a^2)\sin^2(\phi - \theta)}{2(a^2\cos^2\phi + \sin^2\phi)}\right\}[1 + \text{erf}(\lambda)] \quad (4\text{-}1)$$

where $\sigma_Y = a\sigma_X$ (4-2)

$$S^2 = \frac{A^2}{\sigma_X{}^2(1 + a^2)} \quad (4\text{-}3)$$

$$\lambda = \frac{S\sqrt{1 + a^2}\,(a^2 \cos\phi \cos\theta + \sin\phi \sin\theta)}{a\sqrt{2}\sqrt{a^2 \cos^2\phi + \sin^2\phi}} \quad (4\text{-}4)$$

The analysis for this case parallels that given in the preceding paragraphs for the case of correlated signals. Consequently, we will be brief in our discussion for this case.

4.1 Symmetry of Density Function

One can show that

$$P_\phi(-\phi;\ a,\ S^2,\ -\theta) = P_\phi(\phi;\ a,\ S^2,\ \theta) \quad (4\text{-}5)$$

and

$$P_\phi(90° - \phi;\ a,\ S^2,\ 90° - \theta) = P_\phi(90° + \phi;\ a,\ S^2,\ 90° + \theta) \quad (4\text{-}6)$$

This gives symmetry about $\theta = 0°$ and $\theta = 90°$ so that we have

$$\beta(-\theta) = -\beta(\theta) \quad (4\text{-}7)$$

$$\beta(90° + \theta) = -\beta(90° - \theta) \quad (4\text{-}8)$$

and

$$\text{Var}[\phi|-\theta] = \text{Var}[\phi|\theta] \quad (4\text{-}9)$$

$$\text{Var}[90 + \phi|90 + \theta] = \text{Var}[90 - \phi|90 - \theta] \quad (4\text{-}10)$$

For example, the variance of Φ for $\theta = -30°$ equals the variance for $\theta = 30°$, and the variance of Φ for $\theta = 120°$ equals the variance for $\theta = 60°$.

One can also show that

$$P_\phi(\phi + 90°; \frac{1}{a}, S^2, \theta + 90°) = P_\phi(\phi; a, S^2, \theta) \qquad (4\text{-}11)$$

This says that

$$\mathrm{Var}[\Phi|\frac{1}{a}, \theta] = \mathrm{Var}[\Phi|a, \theta - 90°] \qquad (4\text{-}12)$$

For example, the variance of Φ when $a = 2$ and $\theta = 30°$ equals the variance when $a = 0.5$ and $\theta = -60°$ (or $\theta = 60°$).

The above symmetries imply that we need only to consider the density function for $0 \leq \theta \leq 90°$ and $0 \leq a \leq 1$. Considerations for other values of the parameters can be made using the symmetry properties. Thus we will restrict our attention to values of θ in the first quadrant and values of a less than one.

4.2 Special Cases

4.2.1 $S^2 >> 1$

For $S^2 >> 1$ and $|\phi - \theta| < 5°$,

$$\cos(\phi - \theta) \approx 1 \qquad (4\text{-}13)$$

$$\sin(\phi - \theta) \approx \phi - \theta \qquad (4\text{-}14)$$

$$\mathrm{erf}(\lambda) \approx 1 \qquad (4\text{-}15)$$

and

$$P_\phi(\phi; S^2, a, \theta) \approx \frac{1}{\sqrt{2\pi}\sqrt{\dfrac{a^2\cos^2\theta + \sin^2\theta}{S^2(1 + a^2)}}} \times \exp\left\{\frac{-(\phi - \theta)^2}{\dfrac{a^2\cos^2\theta + \sin^2\theta}{S^2(1 + a^2)}}\right\} \qquad (4\text{-}16)$$

In the vicinity of θ, ϕ is approximately Gaussian with mean θ and variance $\frac{a^2 \cos^2\theta + \sin^2\theta}{S^2(1 + a^2)}$. One would thus expect the estimator $\hat{\theta}$ to be unbiased for large SNR. The variance is dependent upon a, S^2, and θ; it is periodic in θ with period π. The maximum variance now occurs at $\theta = \pm 90°$ and the minimum occurs at $\theta = 0°$, $180°$. Note that this is a 45° shift from the case of correlated signals. The variance lies within the range $a^2/S^2(1 + a^2)$ to $1/S^2(1 + a^2)$ for all θ. The width of this range, $(1 - a^2)/S^2(1 + a^2)$, increases with decreasing a. The high SNR asymptote is

$$10 \log_{10} S^2 = -10 \log_{10} \sigma^2 + 10 \log_{10} \left\{\frac{a^2 \cos^2\theta + \sin^2\theta}{1 + a^2}\right\} \quad (4\text{-}17)$$

4.2.2 $S^2 = 0$

For the case of no signal ($S^2 = 0$) the density function becomes

$$P_\phi(\phi;\ 0,\ a,\ \theta) = \frac{a}{2\pi(a^2 \cos^2\phi + \sin^2\phi)} \quad (4\text{-}18)$$

The density function is periodic with period π and has extreme points at $\theta = 0$, $\pm 90°$, $180°$. As $a \to 1$, $P_\phi \to \frac{1}{2\pi}$ which is a uniform distribution with mean θ and variance $\pi^2/3$. As $a \to 0$, the density approaches impulses centered at $\phi = 0$ and $\phi = 180°$. We would again expect the estimator to be biased when the SNR is low. Values of the mean and variance for various values of a are listed in Table 4-1.

4.3 Numerical Evaluations

Equations (2-21) and (2-22) were evaluated numerically for several combinations of parameters. The results of these evaluations are shown in Figures 4-1 to 4-3. Figure 4-1 shows several density functions for $S^2 = 5$ dB and $a = 0.5$. Typical plots of SNR versus variance are shown in Figure 4-2. The variance decreases with increasing SNR.

For SNR greater than 6 dB the variance is maximum for $\theta = 90°$ and minimum for $\theta = 0°$. Near $S^2 = 6$ dB, the variance is nearly independent of θ.

The estimator is biased for all values of θ except $\theta = 0, \pm 90°, 180°$. Figure 4-3 is a plot of the mean versus SNR. The bias increases with decreasing SNR; the limiting values are given in Table 4-1. For $a > 0.1$ the bias is less than 7° and for $a > 0.5$ the bias is less than 5°.

4.4 Analysis of Estimators

The analysis of the estimators given in paragraph 3.5 also applies to this case of unequal variances. The discussion and test described in paragraph 3.6 apply also. For high SNR both methods give equal performance. The high SNR asymptotes and test points P_1 and P_2 are included in Figure 4-2. From these graphs we see that there is again a range in which the better method depends upon n. For example, if $a = 0.8$, we see that method 2 is better for $S^2 < -3$ dB and method 1 is better for $S^2 > -3$ dB if $n > 20$. Again we see that the better method depends upon θ as well as upon a and S^2 when $a < 0.7$. Now for many practical considerations we will only be concerned with values of $a > 0.5$ and $S^2 > -5$ dB. In these situations method 1 would be preferred since it is the better method for most values of θ. Also the observations that in general $\hat{\theta}_2$ will not be distributed on $[\pi - \theta, \pi + \theta]$ and that the bias of $\hat{\theta}_2$ does not decrease with n tend to make method 1 the preferred method.

TABLE 4-1
NO SIGNAL CASE ($S^2 = 0$)

a	θ (Degrees)	E[φ] (Degrees)	Bias (Degrees)	Var[φ] (Rad^2)
0.8	0	0.002	- 0.002	3.343
	15	16.581	- 1.581	3.335
	30	32.681	- 2.681	3.314
	45	48.013	- 3.013	3.286
	60	62.543	- 2.543	3.261
	75	76.441	- 1.442	3.244
	90	89.999	0.001	3.238
0.5	0	0.001	- 0.001	3.656
	15	28.188	-13.188	3.539
	30	49.108	-19.108	3.327
	45	63.436	-18.436	3.159
	60	73.898	-13.898	3.054
	75	82.369	- 7.370	2.998
	90	90.000	0.000	2.981
0.1	0	- 0.002	0.002	4.401
	15	69.535	-54.535	3.138
	30	80.174	-50.174	2.788
	45	84.290	-39.290	2.674
	60	86.697	-26.697	2.625
	75	88.466	-13.466	2.603
	90	90.000	0.000	2.597

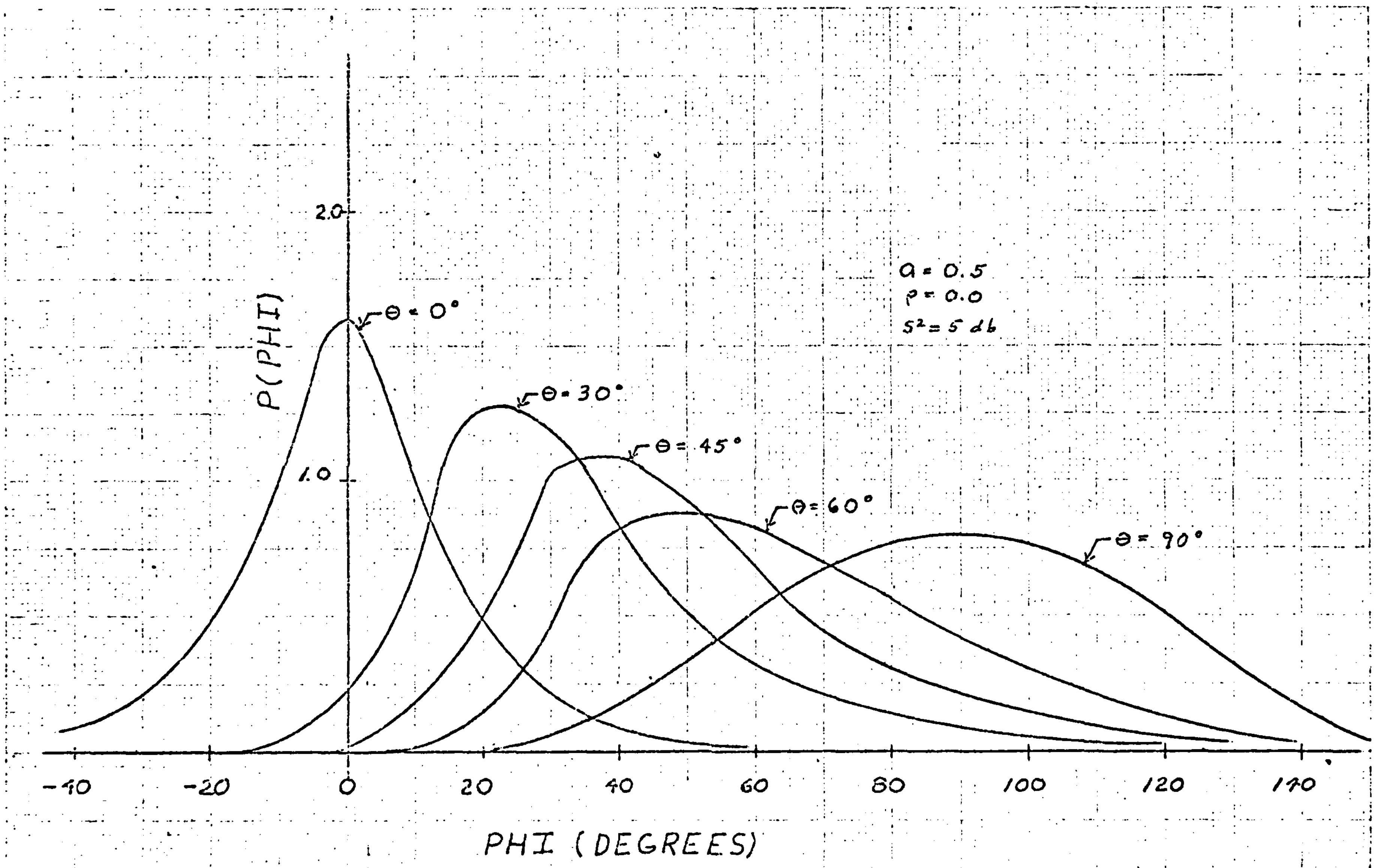

FIGURE 4-1. TYPICAL DENSITY FUNCTIONS

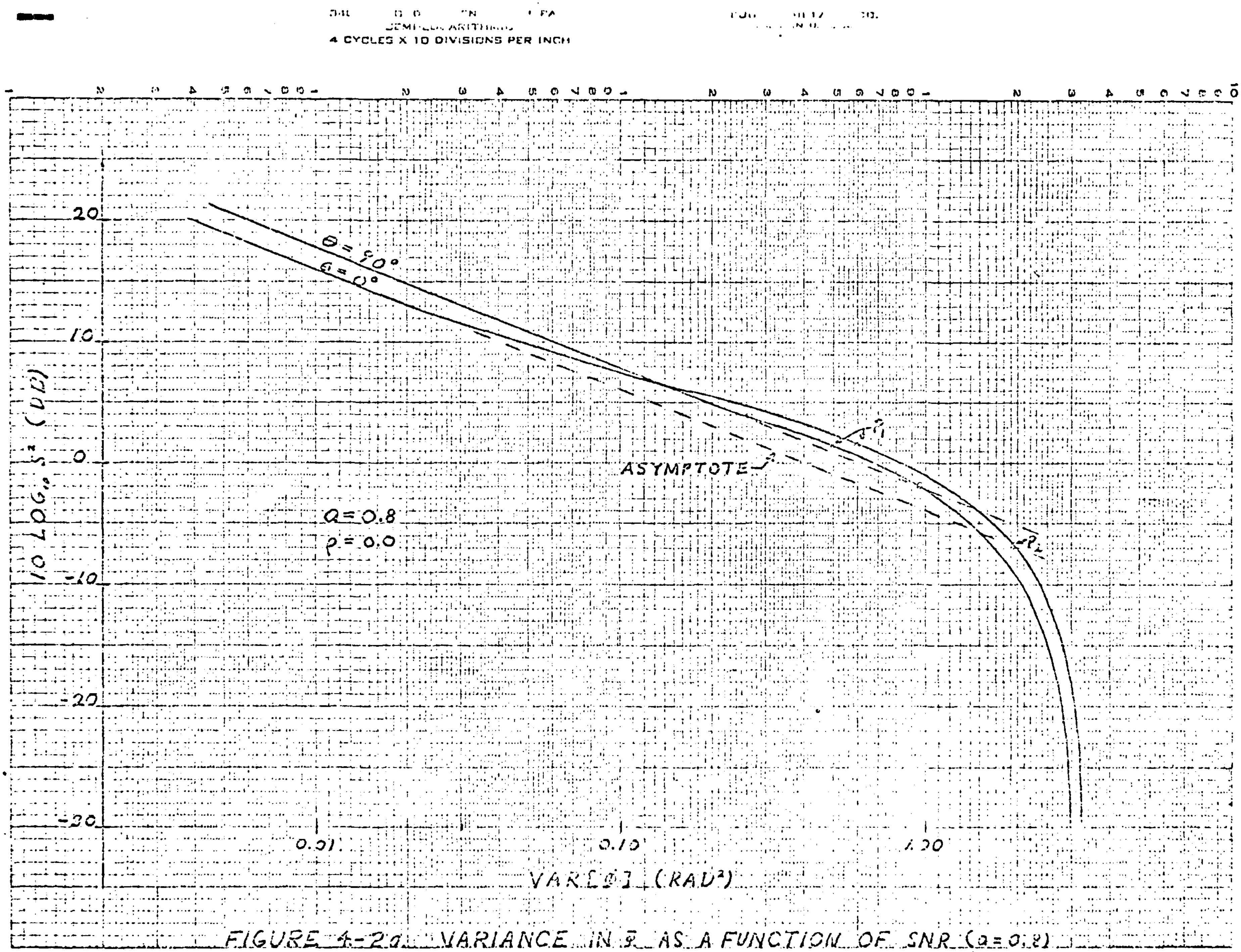

FIGURE 4-2a. VARIANCE IN Φ AS A FUNCTION OF SNR (Q = 0.8)

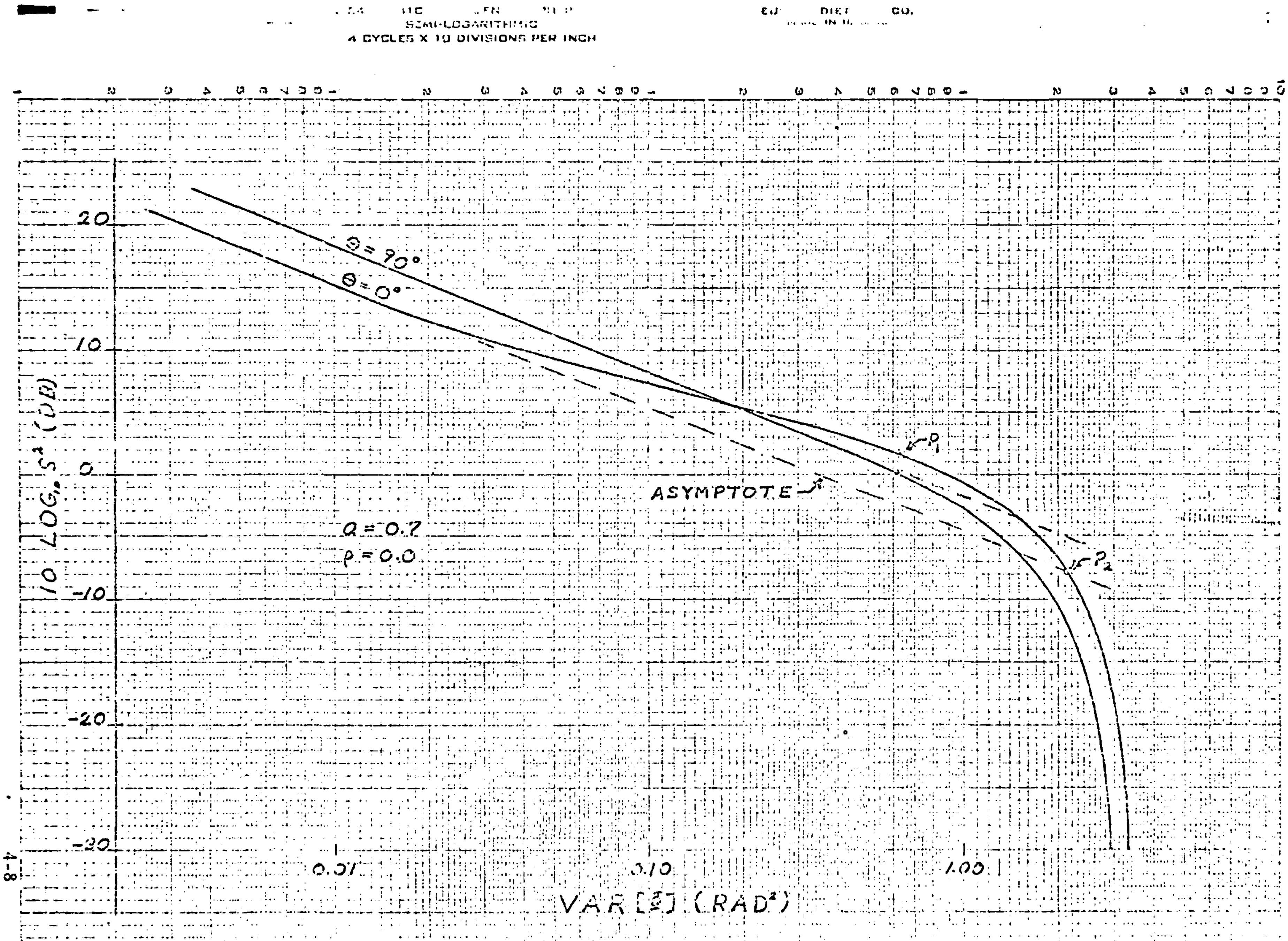

FIGURE 4-25 VARIANCE IN ξ AS A FUNCTION OF SNR (a = 0.7)

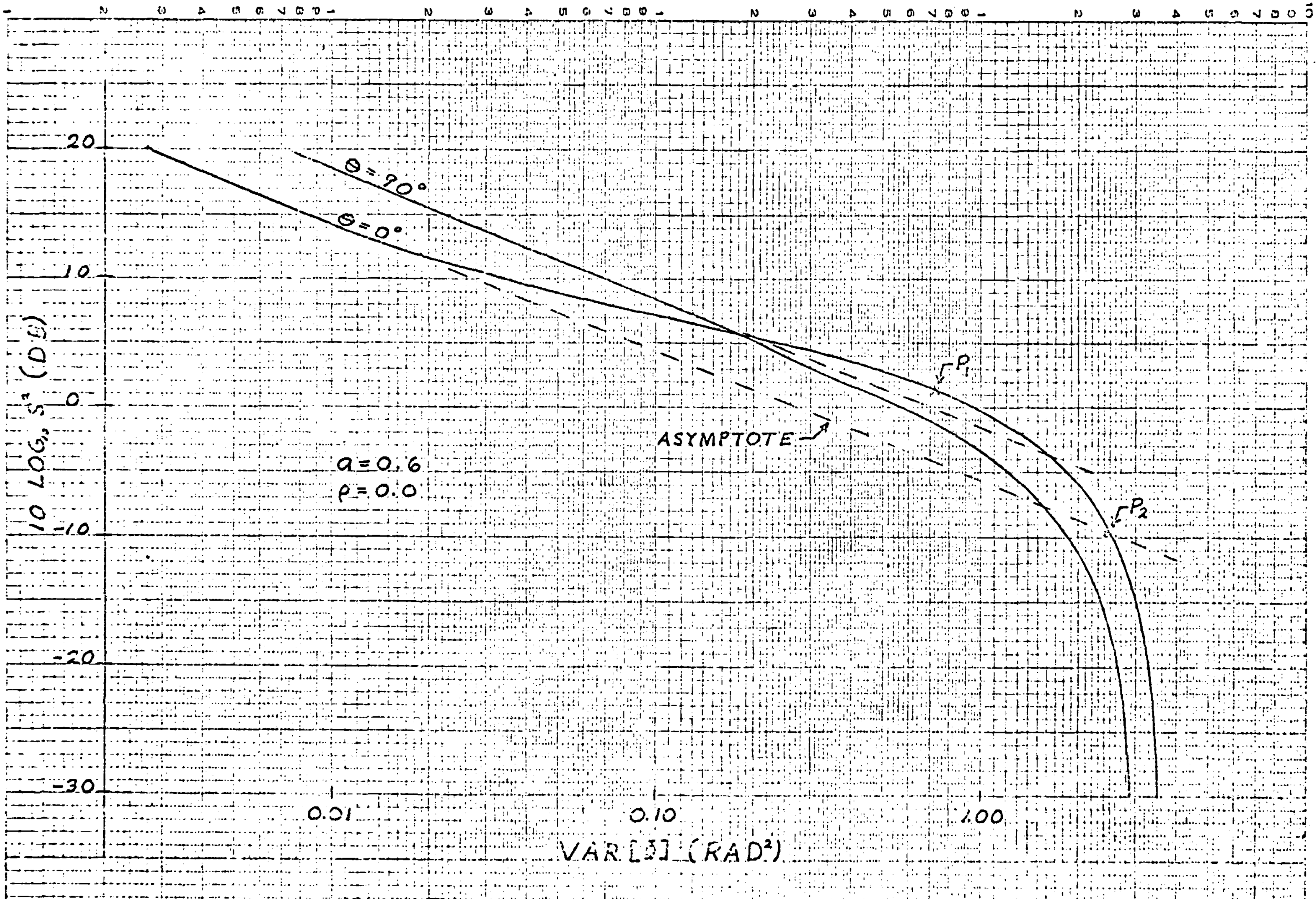

FIGURE 4-2c VARIANCE IN Φ AS A FUNCTION OF SNR (a=0.6)

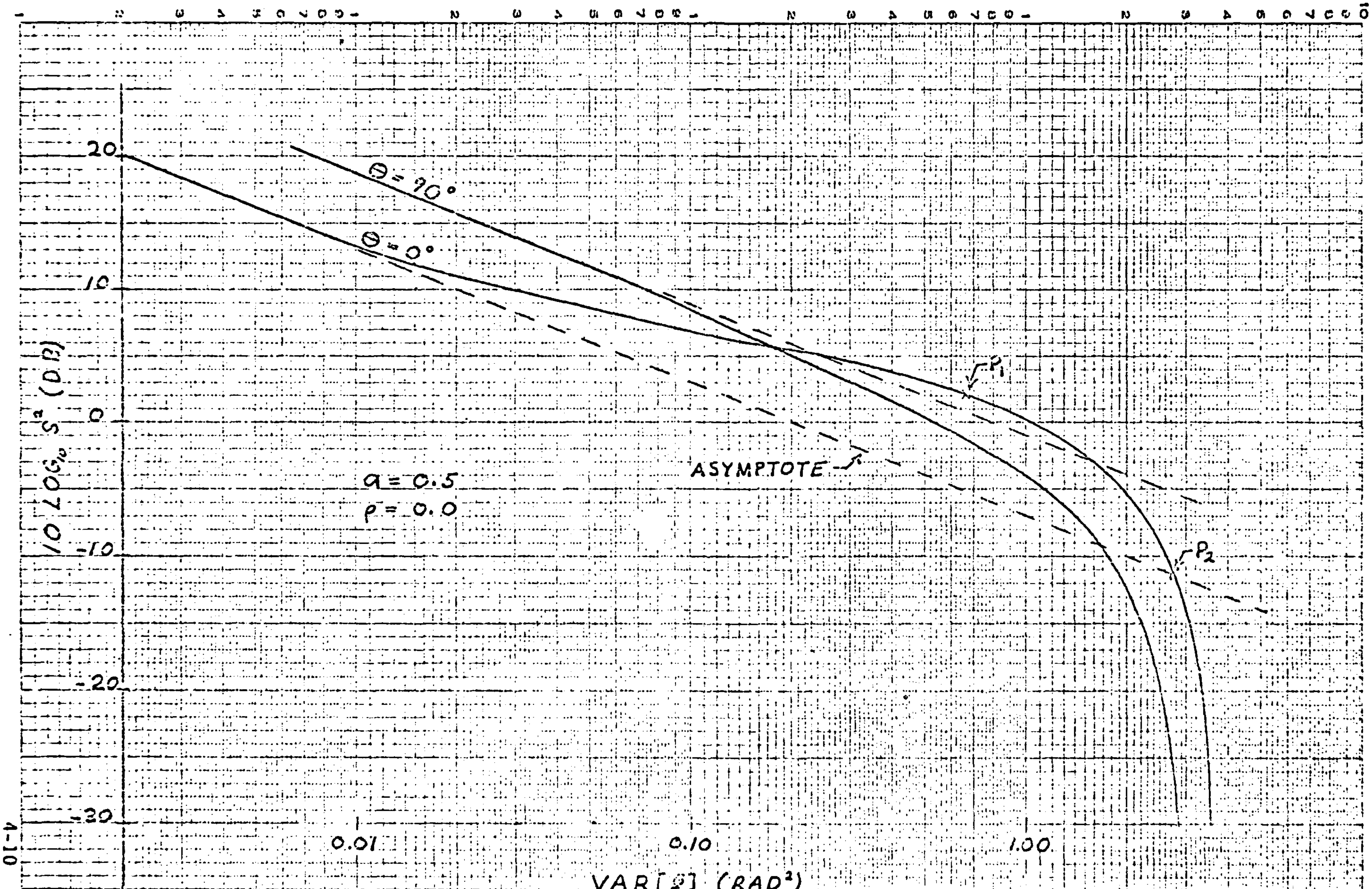

FIGURE 4-2d. VARIANCE IN Φ AS A FUNCTION OF SNR (a=0.5)

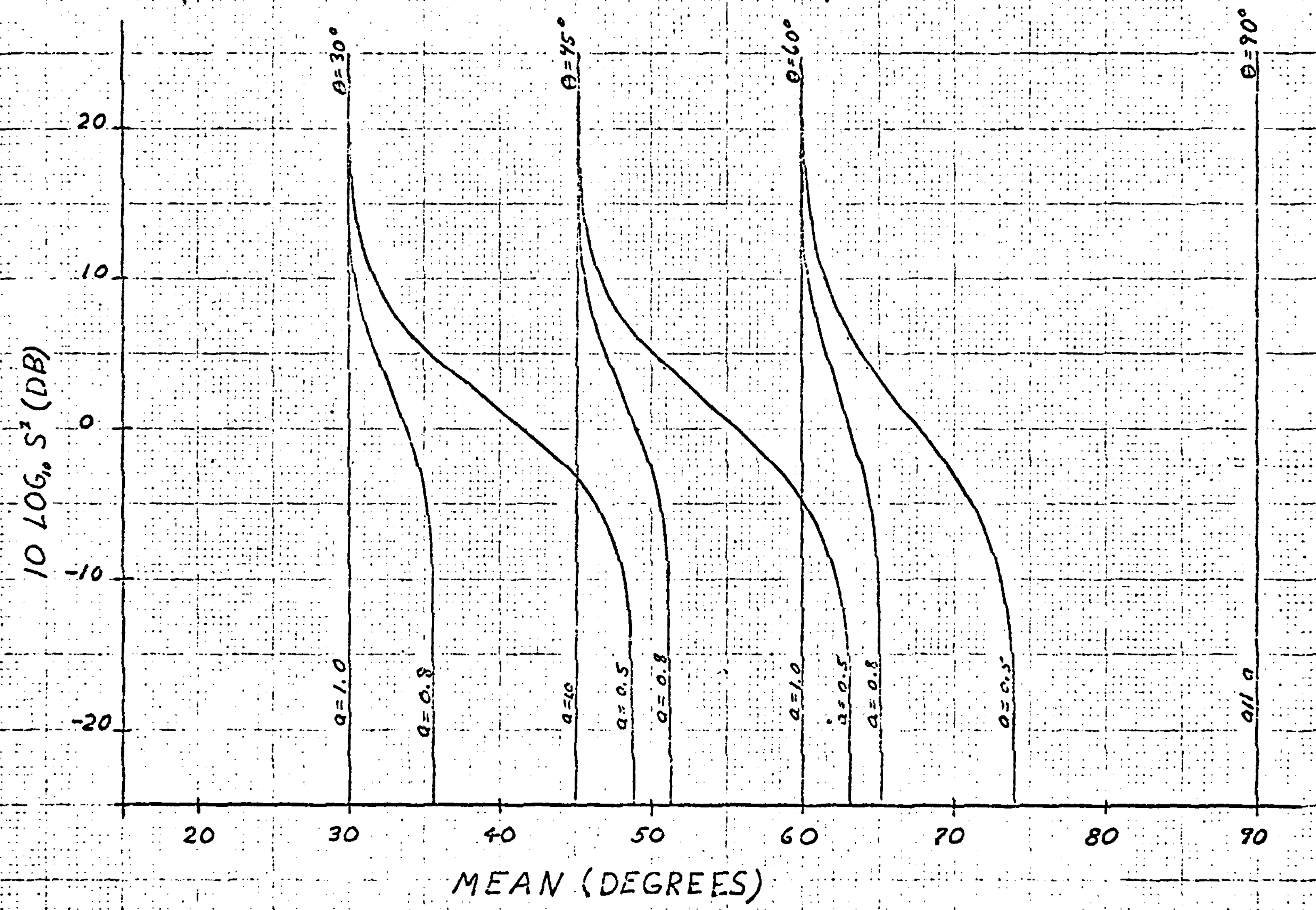

FIGURE 4-3. MEAN OF Φ AS A FUNCTION OF SNR

C

APPENDIX C

SPLIT-BEAM TRACKER ACCURACY ANALYSIS

A theoretical analysis of the performance of a split beam tracker is contained in Reference C-1. One result of that analysis is an expression for the rms pointing error for a linear array of M transducers with independent noise outputs (e.g. half wavelength spacing between elements).

$$\sigma = \frac{4c\sqrt{\pi}}{dM^2\sqrt{T}\cos\theta}\left[\int_0^{\omega_N} d\omega\,\omega^2\,\frac{S^2(\omega)/N^2(\omega)}{1 + M\left[S(\omega)/N(\omega)\right]}\right]^{-\frac{1}{2}} \tag{C-1}$$

where,

where,	d	is the element space
	c	is the propagation velocity
	T	is the system integration time
	M	is the number of sensors
	θ	is the azimuthal arrival angle relative to broadside
	$S(\omega)$	is the signal power spectral density
	$N(\omega)$	is the noise power spectal density
	ω_N	is the input bandwidth processed in radians

For systems not operating at baseband, the integration can be be changed from the range 0 to ω_N to the range $\omega_0 - \left(\omega_N/2\right)$ to $\omega_0 + \left(\omega_N/2\right)$ where ω_0 is the center frequency. We now assume that the signal to noise power spectral density ratio is constant over the processed bandwidth. Denoting this signal-to-noise ratio by γ, the integral is readily evaluated yielding

$$\sigma = \frac{4c\sqrt{\pi}}{dM^2\sqrt{T}\cos\theta}\left[\frac{\gamma^2\left(12\omega_0^2\omega_N + \omega_N^3\right)}{12(1 + M\gamma)}\right]^{-\frac{1}{2}} \tag{C-2}$$

Note that this signal to noise ratio is that defined for an omni-directional sensor.

For narrow band signals (i.e. $\omega_0 \gg \omega_N/\sqrt{12})$

This expression simplifies to:

$$\sigma = \frac{4c}{dM^2\omega_0\gamma}\sqrt{\frac{(1+M\gamma)}{T\omega_N}} \tag{C-3}$$

where θ is assigned near boresight and, therefore, $\cos\theta$ has been approximated as unity.

This expression can be rewritten in terms of the array length L, the wavelength λ, and the bandwidth F, which are defined by

$$L = (M-1)d$$

$$\lambda = 2\pi c/\omega_0$$

$$F = \frac{\omega N}{2\pi} \text{ Hertz}$$

We then have

$$\sigma = \frac{\lambda(M-1)}{\pi L M^2\gamma}\sqrt{\frac{2(1+M\gamma)}{TF}} \tag{C-4}$$

For $M \gg 1$ and $M\gamma \gg 1$, this expression can be simplified further.

$$\sigma = \frac{\lambda}{\pi L}\sqrt{\frac{2}{TFM\gamma}} \tag{C-5}$$

Finally, the ratio λ/L can be related to the beamwidth, B, normal to the linear array (Reference C-2 page 25-40):

$$B_{lin} = 0.88\lambda/L \tag{C-6}$$

This yields

$$\sigma = \frac{B_{lin}}{0.88\pi}\sqrt{\frac{2}{TFM\gamma}} \tag{C-7}$$

Under the assumption that the environmental noise is isotropic and, therefore, independent from sensor to sensor at a 1/2 wavelength spacing, M can be expressed in terms of the broadside beamwidth using Equation C-6:

$$M \approx M-1 = 2L/\lambda = 2\left(\frac{.88}{B_{lin}}\right) \tag{C-8}$$

Substituting into Equation C-7 and evaluating the numerical form results in the desired expression for the rms bearing error obtained with a linear array:

$$\sigma = \frac{0.38\left(B_{lin}\right)^{3/2}}{\sqrt{TF\gamma}} \tag{C-9}$$

We now consider the use of a circular planar rather than a linear array. An approximate expression for the estimation error resulting from the use of a circular planar array can be generated by recognizing that in an infinite ocean of finite depth the vertical beam pattern provides little gain over an omni-directional sensor since both the omni and vertically directional patterns receive energy from the entire water column. In this instance the circular array may be regarded as a line array with a circular aperture weighting function. The result of this "weighting" function is to produce an azimuthal beam pattern with the beamwidth of a linear array having a circular weighting. From Reference C-2, this is

$$B_{circ} \approx \lambda/L = 1.13\ B_{lin} \tag{C-10}$$

Using this result in Equation C-9 yields the error estimate

$$\sigma \approx \frac{0.316\left(B_{circ}\right)^{3/2}}{\sqrt{TF\gamma}} \tag{C-11}$$

In the case of a circular array with a baffle to eliminate noise inputs from the 180° sector behind the array, there is a factor of 2 increase in the effective SNR per element relative to omni-directional noise. As this is the case of actual interest to us, we write

$$\sigma_b = \frac{0.22\left(B_{circ}\right)^{3/2}}{\sqrt{TF\gamma}} \tag{C-12}$$

For convenience, we express both σ_b and B_{circ} in terms of degrees rather than radians by dividing both by $180^\circ/\pi$:

$$\sigma_b^o = \frac{0.03\left(B_{circ}^o\right)^{3/2}}{\sqrt{TF\gamma}} \tag{C-13}$$

We now consider a specific application. Let the integration time T and bandwidth F correspond to a single tone with TF product of one. Then, the rms error becomes

$$\sigma_b^o = \frac{0.03\left(B_{circ}^o\right)^{3/2}}{\sqrt{\gamma}} \tag{C-14}$$

This process is now repeated for a total of N tones. Assuming that these tones are all received independently, we are then averaging over N samples, and the rms error thus decreases as $\sqrt{N}$. Our final expression for the error (in degrees) is then given by

$$\sigma_b^o = \frac{.03\left(B_{circ}^o\right)^{3/2}}{\sqrt{N\gamma}} \tag{C-15}$$

Equation C-14 is plotted in Figure C-1 with $\sigma_b^o\sqrt{N\gamma}$ as a function of B^o_{circ} and in Figure C-2 with σ_b^o as a function of γ, parameterized by N, for B^o_{circ} equal 20^o.

Probably the most significant error in using this result will be due to the omission of SNR improvement that can result from vertical gain due to the general non-uniformity of the noise field in elevation. This can be especially significant at higher frequencies. It should, however, be a useful limit, to the accuracy available from the use of a circular array with a unidirectional illumination pattern.

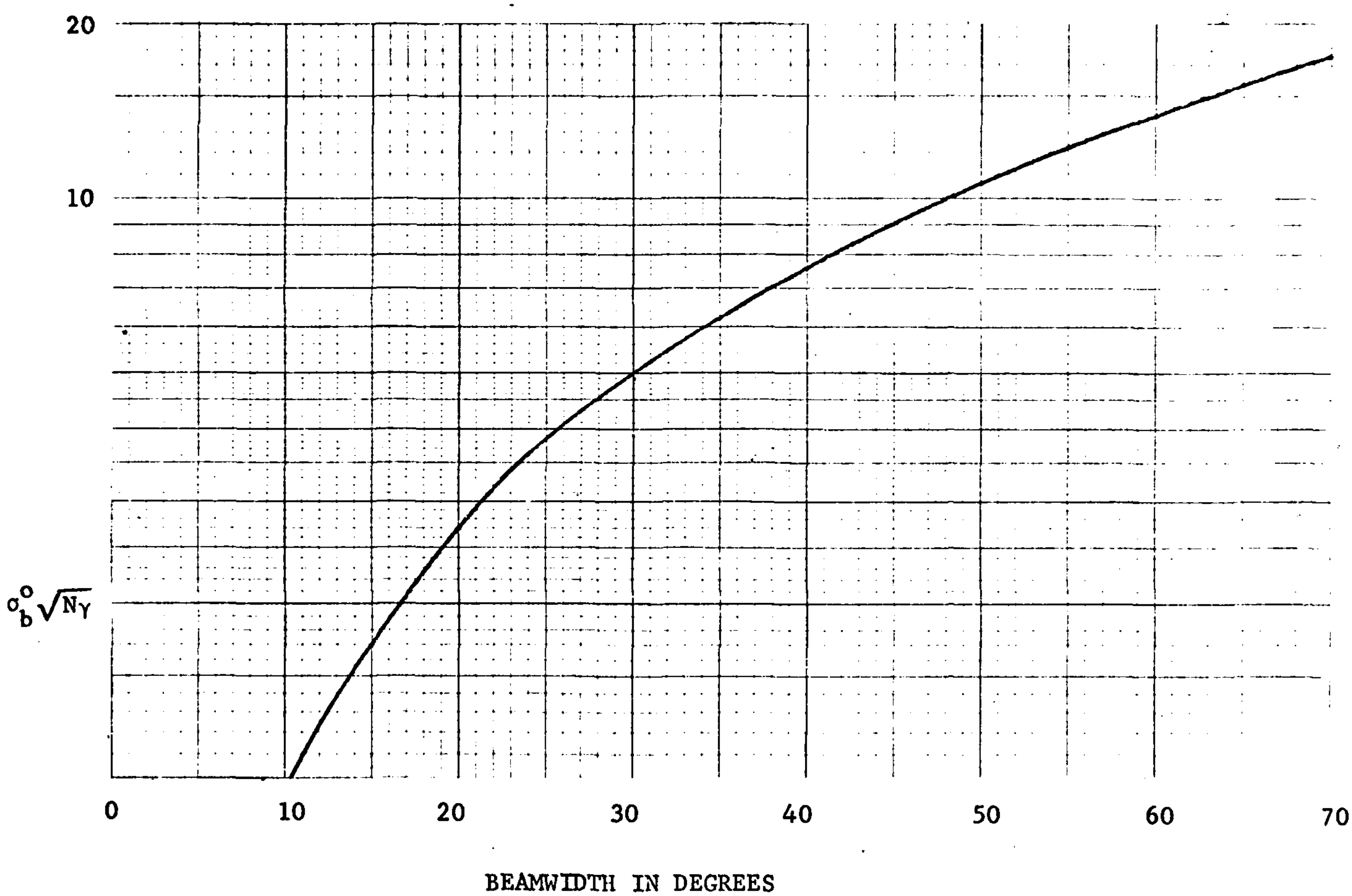

FIGURE C-1 SPLIT BEAM BEARING ESTIMATION ERROR VS BEAMWIDTH

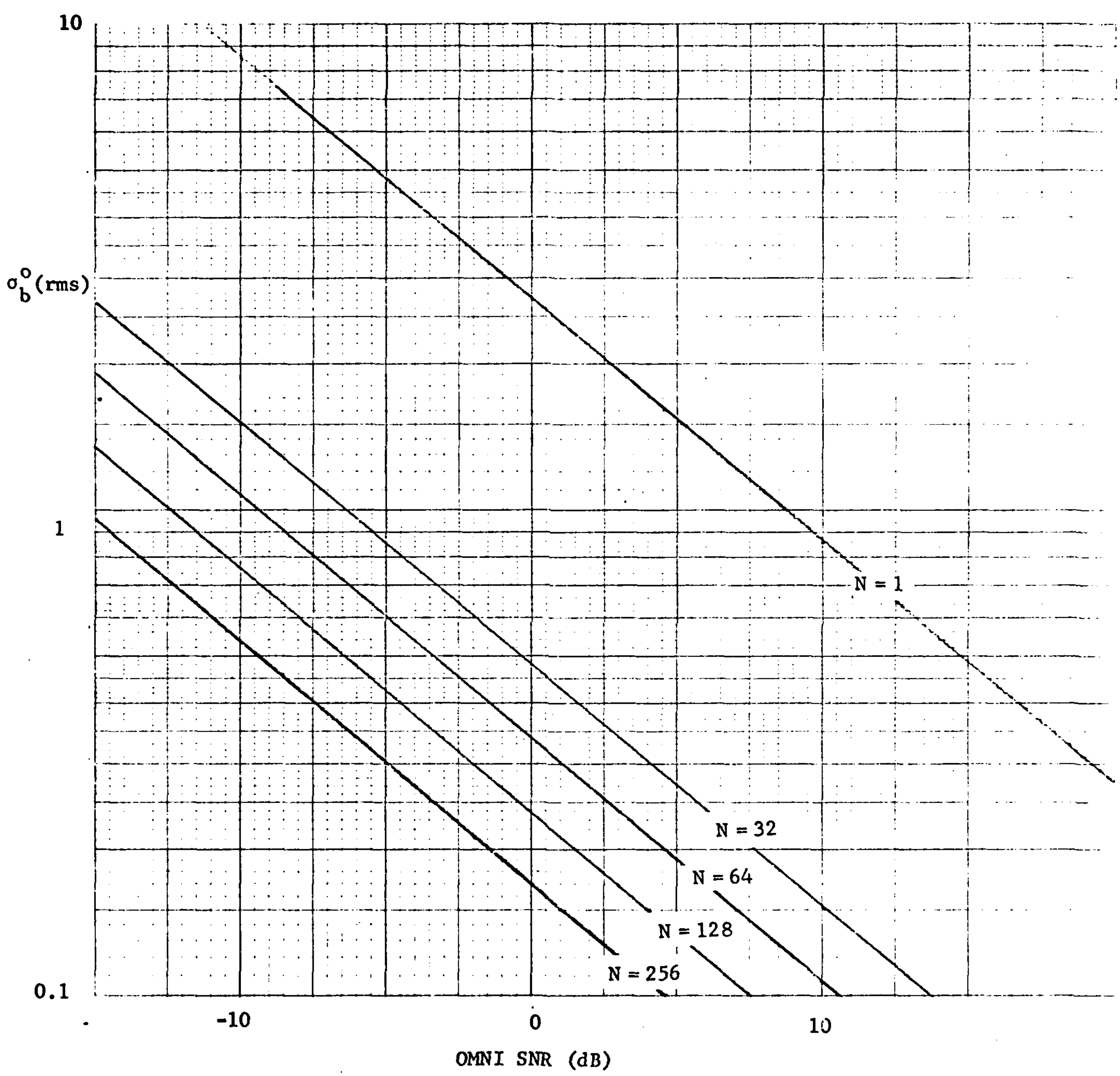

FIGURE C-2 SPLIT-BEAM BEARING ESTIMATION ERROR VS OMNI SNR AND NUMBER OF SAMPLES AVERAGED FOR 20^o BEAMWIDTH

REFERENCES FOR APPENDIX C

(C-1) V.H. MacDonald and P.M. Schultheiss, "Optimum Passive Bearing Estimation in a Spatially Incoherent Noise Field", Journal of the Acoustic Society of America, vol. 46, part I, 1969, pp. 37-43.

(C-2) ITT, Reference Data for Radio Engineers, Fifth Edition, 1968.

D

APPENDIX D

<u>DETECTION OF A MESSAGE REDUNDANTLY TRANSMITTED OVER A FADING CHANNEL</u>

DETECTION OF A MESSAGE REDUNDANTLY TRANSMITTED OVER A FADING CHANNEL

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26 July 1976

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DETECTION OF A MESSAGE REDUNDANTLY TRANSMITTED OVER A FADING CHANNEL

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Technical Note 76-01 26 July 1976

ABSTRACT

This report analyzes the sending of a one-bit message over a Rayleigh fading channel. Transmission of N bits, selected according to a frequency hopped pattern, is analyzed. This choice is based on considerations of the channel itself and a presumed covertness requirement. Performance results, in the form of detection probability curves, are presented. Some new (?) results for calculating and/or inverting the incomplete Gamma ratio are included.

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SECTION 1

INTRODUCTION

The purpose of this report is to present an approach to solving a communications problem, explain the reasons underlying this approach, describe briefly the pertinent mathematical analyses and summarize the results as a set of representative performance curves. We consider here the problem in general; application to a specific problem follows readily from these results.

The problem consists of devising and analyzing a system for transmitting a message. This message is equivalent to one bit of information - e.g., it might be an "Alert", "Ready", or "Acknowledge" signal. The message is seldom sent. When it is, it is a time unknown to the receiver, although the allowable transmission times may be discrete instants, e.g., every minute. In order to receive any such message, the receiver must continually "look" for the alert message. We then have a problem analogous to the classical radar detection problem, wherein performance is determined by the probability of detection versus the probability of false-alarm.

We specify two significant constraints implicit in the design of the transmission system. One is the nature of the communications channel. For this, we assume a fading channel with considerable multipath spread and frequency smear. Further, the nominal time delay and frequency shift between the transmitter and receiver, although bounded, is unknown. Specifically, this might be an underwater acoustic communications channel with moving transmitter and receiver. The second constraint concerns the possible existence of unauthorized receivers, which we call interceptors. This means that we wish to structure the message such

that it is recognizable to the receiver but not (or, more exactly, to a much lesser extent) to the interceptor.

The above considerations lead to a redundant message structure. Although the message has information content of only one bit, we picture it as consisting of a collection of transmissions, each individual transmission being a pulsed tone. The tones are scattered throughout a region of time and frequency according to a pattern known both to the transmitter and intended receiver. This may be viewed as providing time and/or frequency diversity to partially compensate for the fading channel. Alternatively, it may be viewed as allowing reliable yet covert communication in that it is assumed that the tone structure is unknown to a would-be interceptor. Both interpretations are correct in that the redundant pulsed tones may be transmitted with less total signal energy than would be required for a single pulsed tone for the same probability of detection.

First we discuss the tone locations in time and frequency and the effect that various patterns have on interceptability.* Then, we present briefly the type of processing required at the receiver. We also consider here the multipath spread and Doppler smear characteristics of the channel. Finally, we examine the anticipated system performance, as determined by the total number of tones comprising the message and the received signal-to-noise ratio per tone.

*A detailed analysis of the performance realized to an interceptor shall be presented in a companion report.

SECTION 2

ALERT MESSAGE STRUCTURE

2.1 THE TIME/FREQUENCY MATRIX

Throughout this report we shall utilize a two-dimensional (time and frequency) representation for the alert message. The time dimension will be divided into intervals of length T, referred to as "time-slices"; frequency will be divided into intervals of size Δf, referred to as "frequency-cells." For simplicity, we shall set $(\Delta f)\ T$ to unity. The intersection of a time slice and a frequency cell will be referred to as a tone-location or a tone. The intersection of N continuous slices and M cells constitutes a "time-frequency matrix" of size N x M.* An example of such a matrix is shown in Figure 2-1. We will denote a matrix of this form by S and a tone within the matrix by $S(i,j)$. This definition of S is thus far independent of absolute time and frequency values. We now restrict our attention to an arbitrary but fixed range of frequencies of total width F. Time is considered to be an infinite continuum quantized into increments of size T. Thus, each time quantum defines the beginning of a different time/frequency matrix. Where necessary, this will be denoted as S_k where $t_k = kT$ represents the absolute time value corresponding to the index $i = 1$.

*Note: whenever indices, subscripts, etc. are used, the first term will specify time; the second, frequency.

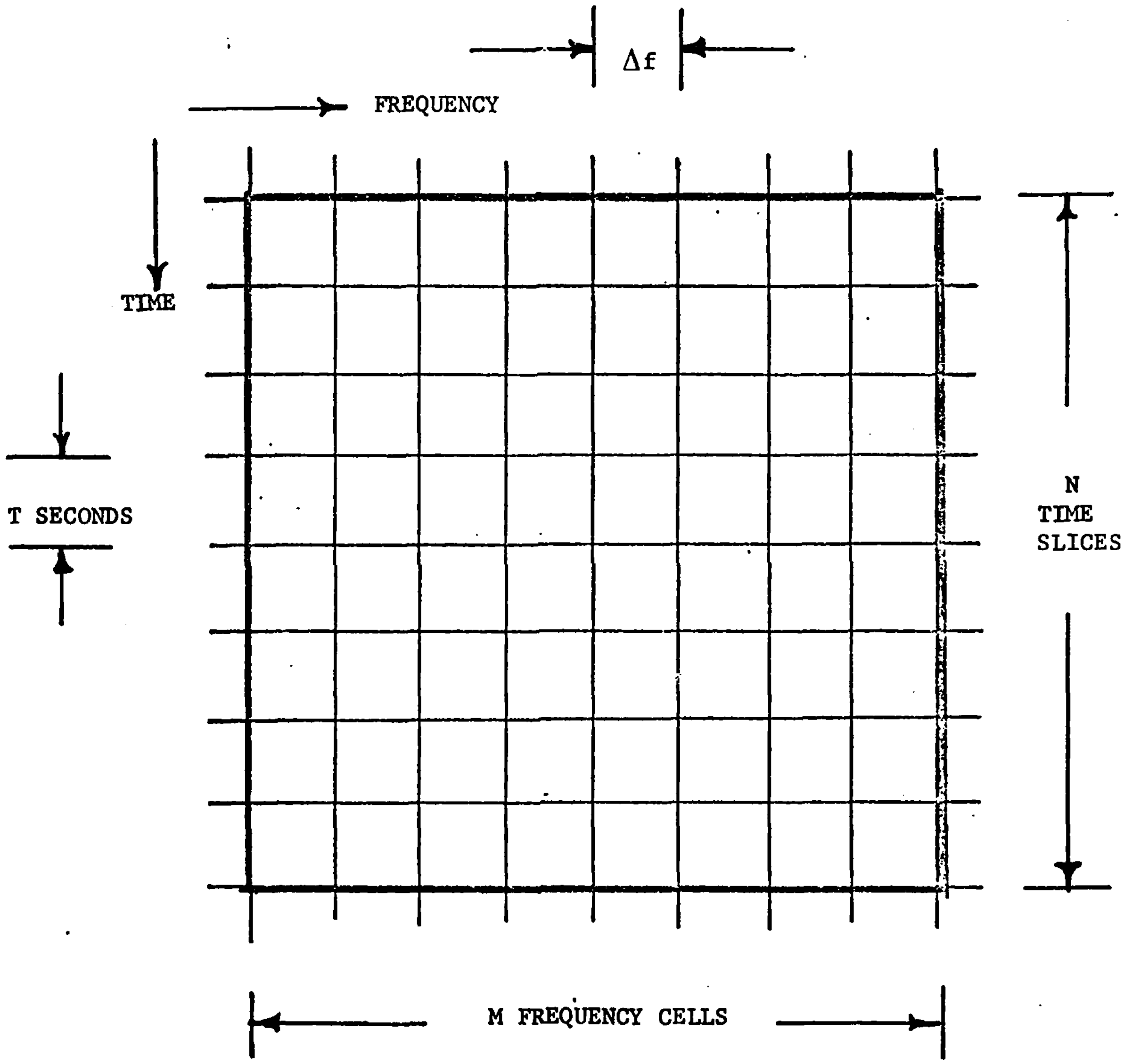

FIGURE 2-1 TIME/FREQUENCY MATRIX, N=M=8

2.2 BASIC CONSIDERATIONS

We define the alert message in terms of a time/frequency matrix of size N x M, where N and M are to be chosen as part of the actual signal design. The message itself is defined as a pattern of tones within this matrix S. Each time slice of S determines a transmission; within each slice, the particular tones (one or more) dictated by the pattern are activated. Thus, the requisite transmitter for this system is similar to that which would be used for multi-tone On/Off keying (MTOOK). For purposes of analysis, the time origin of the message is irrelevant. The ordering of active tones throughout the time/frequency matrix, known as the pattern or key to the message, is all that is important.

It was earlier stated that the channel was subject to fading. We note here that the delay and frequency spread are such that, in general, coherent combination of the received tones is impossible. Thus, incoherent combination must be used. Let the total number of tones in the pattern be L. As long as the key is known to both the transmitter and receiver, and the time/frequency matrix is structured so that each cell is subject to independent Rayleigh fading, the actual location of these L tones within the N x M matrix does not matter. That is, system performance will be dictated purely by the total number of tones and the total transmitted signal energy. The assumed independent fading is realistic, especially if the individual tones are sufficiently scattered in time and/or frequency. If not, if there is correlated fading among the tones in the pattern, full diversity effects will not be realized and performance will be somewhat degraded compared to the idealized case.

Scattering of the individual message tones in time and frequency is in basic agreement with our desire to make the signal

non-detectable to an unauthorized receiver. Intuitively, it is clear that the greater the time/frequency space within which the L tones are scattered and the less the clustering of tones within that space, the less detectable the message will be to someone not possessing the key. We assume that the pattern chosen is sufficiently diverse to ensure independent fading regardless of its detailed composition. We then more finely define the message key based on considerations of interceptability.

We do not intend here to derive expressions for intercept probabilities, rather, we wish merely to derive a general conclusion concerning the form of the pattern such that the probability of intercept will be minimized. Although the interceptor does not know the pattern we will assume that it has perfect knowledge of the time/frequency matrix - i.e., it knows T, Δf, N and M and is synchronized in both time and frequency.* Then, for each time slice it receives, it may separately detect the energy in each frequency cell. Alternatively it may detect the energy contained in the total band.

Whether narrow or wide-band detection is used, we will assume for present purposes that the interceptor examines only one time slice at a time, each slice independent of all others. One might identify more sophisticated processing schemes for the interceptor wherein the entire matrix is examined. These will be examined in a companion report. Further, we note that the intercept probability for many schemes of this

*The assumption of synchronization is not unwarranted; an approximation to this may be realized by multiple overlapped intercept receivers.

form will be directly and monotonically related to the intercept probability for one time slice. Thus, it is meaningful to structure the message to minimize the basic intercept probability P_I for one slice.

Our initial discussion showed the detection probability to be a function of the total signal energy E and the number of tones L in the message. This fixes the energy per transmitted tone at E/L. Given this constraint, it is clear that the least interceptable signal is one containing an average of one or fewer tones per time slice. To see this, consider briefly two ways in which an unauthorized receiver might process each time slice. If the interceptor employs narrow-band analysis, any active tone will be equally detectable with probability P; the probability of detecting one or more out of n tones will be of the form $1 - (1 - P)^n$ which increases with n. For wide-band analysis, increasing the number of tones increases (linearly) the total energy in the band - once again, P_I increases monotonically with the number of tones. Thus our conclusion.*

The general nature of the above results leads to the somewhat heuristic conclusion that the alert message should be structured such that in any time slice, at most one tone is active. Just as examination of one time slice (all the tones in one row) of the matrix was postulated for the interceptor, so too could we consider examination of one frequency cell (all the tones in one column). This leads to a conclusion which is the logical dual of the above - in any frequency cell, at most one tone should be active.**

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*If one assumes a total transmitted energy per time slice regardless of the number of active tones, a different result will follow (Reference 2). This does not apply to our problem.

**Subject to practical limitations as discussed later.

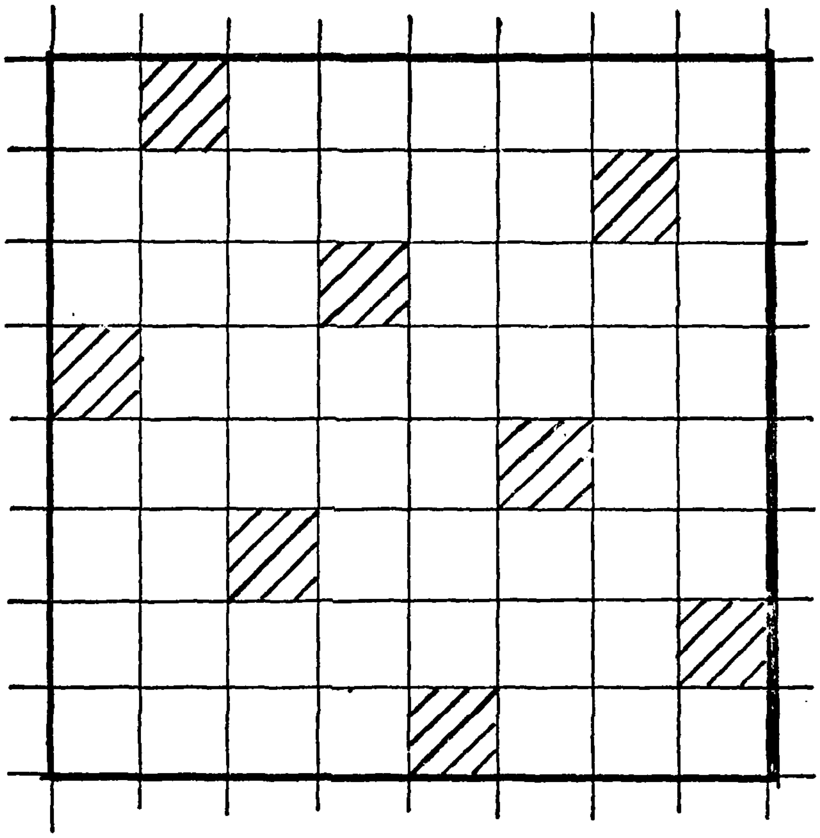

FIGURE 2-2 ILLUSTRATIVE MESSAGE PATTERN, N = M = 8

Combining these criteria, we require a message pattern consisting of at most one active tone in any row or column of the matrix. This is conceptually equivalent to the problem of placing rooks on a chessboard such that no rook can capture any other. Many solutions are possible - an example is shown in Figure 2-2.

The above result is certainly not surprising. It simply represents the greatest extent of scattering of the tones in the allowed region. It leads to the obvious conclusion that either dimension of the matrix must be at least equal to the number of tones in the message,

$$N \geq L \quad ; \quad M \geq L \tag{2-1}$$

Having an inequality in Equation (2-1) is equivalent to having a totally unutilized time slice or frequency cell. This spreads the message over an even greater time/frequency area. Although this might decrease interceptability when more sophisticated techniques are employed, it also increases the operational requirements for the intended receiver. We will assume that our message is sufficiently diverse and that we do not wish to make the message length or bandwidth any larger than is necessary. We then have

$$L = N = M \tag{2-2}$$

which we will assume to be true in the rest of this report. The symbol N will interchangeably be used to denote the number of tones in the message or the time or frequency dimension of S.

It should be noted that the above arguments tacitly assume that a message structure as defined by Equation (2-2) or (2-3) will not be of excessive duration or bandwidth. If the number of transmitted tones required to yield a desired level of performance is too large, then

more than one active tone must be put into some rows or columns of the matrix. As the communications system, especially for some of the channels under consideration, is bandwidth limited, we "double-up" in time. That is, we hold to only one tone per time slice but allow more than one tone per frequency cell. Presumably, the message might be spread in time across n matrices, each of size N x M. Each submatrix would be as discussed in this section, the composite matrix would be of size nN x M and the message would contain nN tones. Our formulations for system performance will not be affected by this; one might wish to modify the expressions for P_I to account for the extra detectability introduced by this modified structuring. Under most conditions, however, enough bandwidth should be available to accommodate the pattern within one N x M matrix, and such refinements will not be needed.

SECTION 3

PROCESSING THE RECEIVED SIGNAL

In this section we describe the form of processing used at the receiver to determine whether a message is present or not. The implications of the time and frequency spreading and shifting associated with the channel are also discussed.

3.1 THE RECEIVER

The receiver constructs time/frequency matrices identical to that determining the transmitted signal. One matrix is formed every T seconds. The individual tone cells are of duration T and width $\Delta f = 1/T$ as discussed earlier. In order to do this, input samples must be accumulated at a rate M/T; these are then processed via an FFT to produce M frequency cells for one time slice. Each time $(t_k^{'})$ defines a new matrix S_k consisting of the N most recent time slices.

From each matrix, the tones corresponding to the predefined key are extracted and square-law combined to obtain a decision variable which is compared to a threshold level. This represents optimum (maximum likelihood) processing for independent Rayleigh fading signals in the presence of white Gaussian noise. The processing operations are illustrated in Figure 3-1, wherein the time/frequency matrix is depicted as M shift registers in parallel, each of length N. The post-detection shift register processing produces an output which corresponds to the cross-correlation of the message key with the detected power level in the resolution cells.

Let S_k be the matrix containing all N received message tones. Although S_{k-1} and S_{k+1} each contain N-1 of the N transmitted tones, they

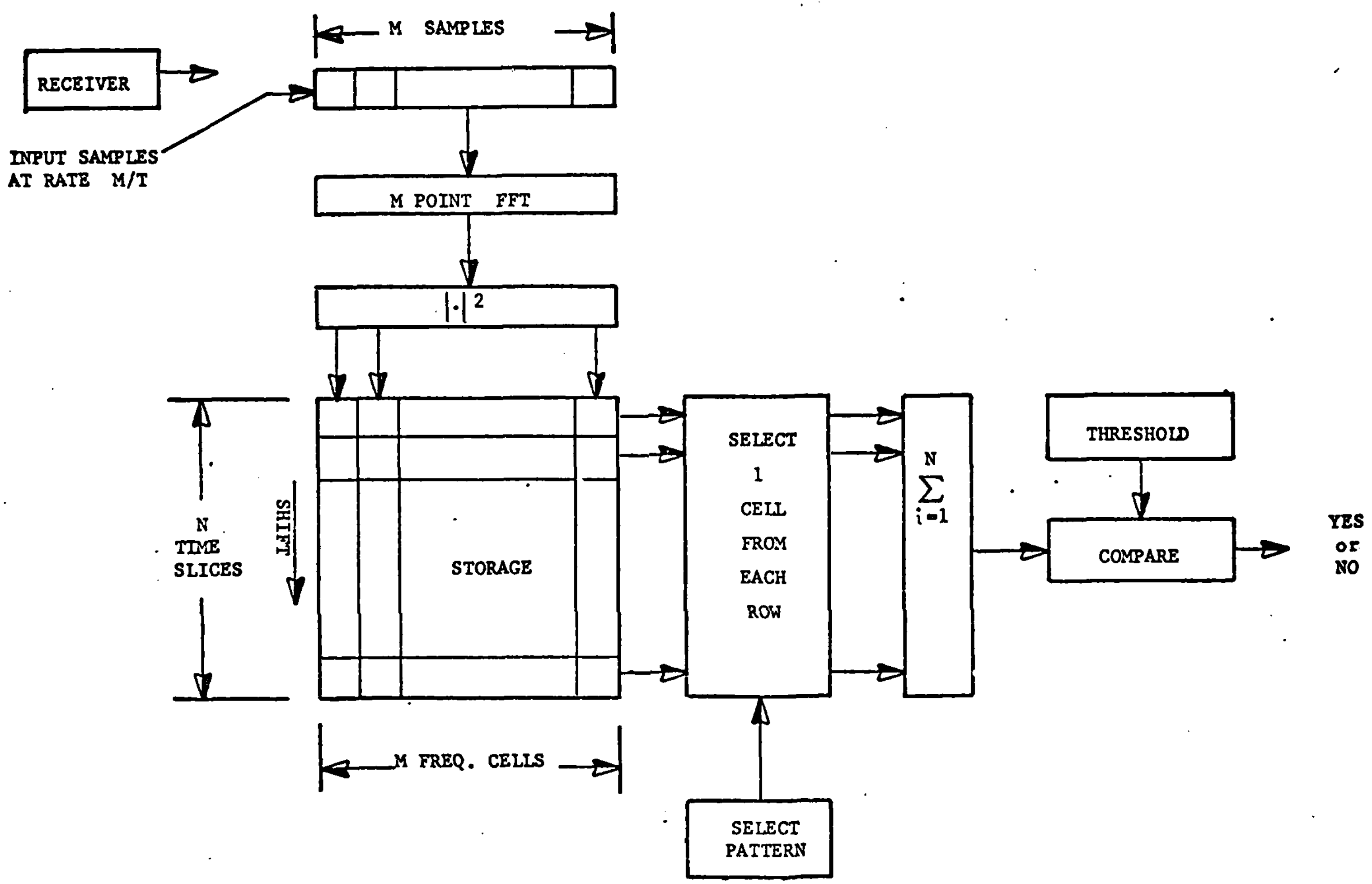

FIGURE 3-1 RECEIVER BLOCK DIAGRAM

are out of alignment with the pattern. The particular form of pattern specified in the previous section is such that any displacement from perfect alignment results in square-law combining the envelopes of N resolution cells, none of which contain signal. Thus, the tones selected according to the key from a received matrix ideally either all contain a signal (and this happens for only one time t_k), or are all purely noise. Thus, the correlation function formed by the receiver is ideally a delta function when a message is present.

In actuality, there will be spillover both in time and frequency which will broaden the correlation function. This results both from the spreading and the shifting of the individual tones comprising the message. Nonetheless, the receiver responses at T seconds before and after the peak are likely to be small compared to the response at the peak. Thus, the correlation function, although not an impulse, will still be quite narrow. The significance of this is that, in order to have a reasonable probability of detection, the correlation processing depicted in Figure 3-1 must be performed every time slice.

A consequence of the sharpness of the receiver correlation function output when a message is present is worthy of note. Our analysis will assume a stationary noise environment. Then, when no signal is present, the decision variables calculated from successive matrices S_k and S_{k+1} will be independent. If the threshold is exceeded at time t_k, it most likely will not be exceeded at either t_{k-1} or t_{k+1}. That is, at t_k, the correlation function will be similar in shape to that when a signal is present (it will have a sharp peak) and will be indistinguishable from that due to a true message. If, however, the noise environment is

non-stationary, or if there is interference, then the noise-only outputs will be correlated and several successive time instants may result in threshold exceedances. Separately considered, these would each be false alarms. Considered as a whole, they form too broad a peak for the correlation function and can be rejected as being non-signal in origin. Thus, by a small amount of extra processing, one can improve performance when non-stationary noise or interference is present. In effect, one is correlating the correlation function to reject a certain class of false alarm.

3.2 EFFECTS OF THE CHANNEL

In this section we describe qualitatively the changes imparted to the individual message tones by the channel. The receiver processing, described in the previous section and analyzed later, assumes these to be negligible. Actually, these effects can be significant, and the idealized results should be modified to account for these disturbances. The significant impairments to the message due to the channel are described in Section 3.2.1 and 3.2.2. In Section 3.2.3 we then discuss briefly how to modify the ideal results to incorporate these effects for analysis of an actual system.

3.2.1 Spreading

The original transmitted tones have a specified duration and frequency width. These are both increased due to passage through the channel. This results in spillover of signal energy into adjacent tone locations in the received time/frequency matrix. As the matrix is sparsely inhabited, this is not as significant a problem as it might be in a conventional multitone communications system. However, as the received signal energy is spread over a greater time/frequency area, less signal energy is contained within the nominal tone location in the matrix. This is effectively modeled as a decrease in the received signal-to-noise level per tone.

3.2.2 Shifting

The alert message is defined by a pattern. This pattern is relative---it is made absolute by choosing an origin for it in the time/frequency plane. The message is received with this origin shifted The

time shift is caused by propagation delay; the frequency shift, by the effects of Doppler. We assume that the amount of these shifts are unknown, but that bounds for them are available. The receiver must then be able to detect the message within some rectangular region of uncertainty in time/frequency centered about a nominal origin.

Since the signal is sent at a time unknown to the receiver, the nominal origin in time of the received signal is unknown. Thus, shifting this origin in time has no effect on the receiver as it must search through all time for the message anyway. We note here that the only instance in which this time shifting is significant is when the propagation path fluctuates so rapidly that different time slices in the message are delayed by different amounts. We assume that this does not happen---i.e., there is not "time distortion."*

In order to avoid searching throughout the frequency, as well as the time continuum, it is assumed that the frequency origin of the message is fixed and known. Then, ideally, the receiver need look only for the pattern as defined by this nominal frequency value.

In actuality, the message is received in a frequency band related to the nominal band by a shifting up or down in frequency and a simultaneous stretching or compressing. If the relative velocity between the transmitter and receiver were known, then the Doppler offset (as a function of frequency) would be known and the pattern of the received message tones could be determined from the nominal pattern. The receiver would then merely examine the time/frequency matrix for a

*In the design of an actual system, one has to consider the change in path length from message start to finish caused by motion of the transmitter relative to the receiver. This may not be so readily dismissed.

message defined by this distorted pattern.

We assume that this knowledge is not available---all that is known is a maximum possible relative velocity, $\pm V_{MAX}$. This in turn specifies a maximum possible frequency shift. Let the highest transmitted frequency be f_{MAX}. Then, the maximum shift is imparted to this tone and is bounded by

$$\pm \Delta F = \frac{\pm V_{MAX}}{c} \quad f_{MAX} \qquad (3\text{-}1)$$

where c is the propagation velocity. When this is large enough to cause the received tone corresponding to f_{MAX} to lie in one of several possible frequency cells, examination in the receiver of multiple Doppler hypotheses is necessary in order to ensure detection of the message.

As the Doppler shift is not constant across the band of frequencies comprising the message, formation of these multiple hypotheses is not a straightforward procedure. The optimum hypothesis test involves the use of a separate sampling frequency in the receiver processing for each postulated value of (V/c). The processing burden and memory requirements (storage of NM terms for each hypothesis) presumably render this approach unfeasible. A cruder but simpler method is to compute the received time and frequency of each tone under a postulated value V/c and to select the nearest time/frequency cell from the nominal matrix. This results in sampling, transforming, and storing the incoming signal once and then performing multiple correlations on it. Each correlation process corresponds to a particular hypothesized distortion of the nominal message pattern.

The number of such correlations required may be equated to the number of cells in which the highest tone can be received. From equation (3-1), this is roughly

$$\#\ \text{hypotheses} \simeq (2\ \Delta F/\Delta f) \qquad (3\text{-}2)$$

More exactly, the nominal pattern will always be examined. As $|\Delta F|$ increases, an equal number of patterns representing positive and negative values of ΔF will be required. Then,

$$\# \text{ hypotheses} = 1 + 2 \left\lfloor \frac{\Delta F}{\Delta f} + \frac{1}{2} \right\rfloor \qquad (3\text{-}3)$$

where $\lfloor x \rfloor$ is the "greatest integer less than or equal to" function.

A related problem due to shifting is lack of synchronization. As the message appears only once, there is no chance for the receiver to adaptively change its timing and sampling rate to establish sync. Thus, a received tone may straddle two time slices. In the presence of Doppler it may also straddle two frequency cells. This would be true even if there were no spreading in time or frequency. Thus, only part of the available signal energy is used when a particular time/frequency cell is examined.

This could be partially alleviated by finer quantizing in time and/or frequency---i.e., rather than rounding a postulated received frequency to the nearest Δf, the response at the nearest $(\Delta f/2)$, etc., could be used. The additional resolution in frequency is easily obtained by means of decimated FFT processing. An equivalent increase in time resolution can be provided by FFT overlap. Doubling the resolution in each dimension increases the FFT processing by slightly more than a factor of four and the required buffer space by a factor of four. Whether this processing burden is warranted in order to achieve the corresponding performance gain (less than 2dB) depends on the performance/cost tradeoffs for the particular problem at hand. Similarly, even finer resolution could be obtained, but one rapidly reaches the point of vanishing returns.

3.2.3 Interpretation of Results

In Section 4 we will derive results applicable to a non-dispersive fading channel and a receiver which is synchronized to the received message tones. To interpret these results in a more realistic setting, modifications, as indicated in the discussion of spreading and shifting, have to be made. The form of the given results remains valid, but the various parameters in terms of which these results are formulated need to be adjusted.

One necessary modification results from the multiple correlations required to test the various possible Doppler hypotheses. Let the number of such be k. We stated earlier that the receiver processed the incoming data once every T seconds. This is still true, but now, k rather than 1 is the number of decisions made every T seconds. Thus, the effective time interval between tests is T/k. A common measure of performance for a detection system is the false-alarm interval, which we define as the time interval during which the probability of having one or more false alarms is one half. This is a function of the false alarm probability per decision and the rate at which the decisions are made. When using the performance curves given in Section 4, it should be remembered that P_{FA} is defined per decision, not per time slice. The false alarm interval is then determined from Table 4-1 by using the value of T/k, not T.*

The number of multiple hypotheses required clearly depends on the choice of tone width. However, the false alarm interval is essentially

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*To be exact, the k multiply formed decision variables are not independent. Thus, use of T/k is a worst case adjustment. Dividing T by only one-half of k might be more realistic.

independent of this. To see this, let k be given by equation (3-2). As the time/frequency matrix is constructed with $\Delta f = 1/T$, we alternatively have

$$k \approx (2\Delta F)T$$

and the effective time interval between decisions is

$$\frac{T}{k} \approx \frac{1}{2\Delta F} \tag{3-4}$$

The choice of basic tone size thus has little impact on the system operating characteristics as determined by the false alarm interval. It might seem that the number of multiple hypotheses would be a measure of the requisite complexity of the receiver. More significant is the rate at which these decisions must be made, which determines the required processing capability. From equation (3-4), this too is independent of the basic tone structure.

We now consider spreading and lack of synchronization. Although different phenomena, they are both manifested by a decrease in the signal energy in the time/frequency cell examined. The process discussed to counter an unknown Doppler shift is such that the center (in time and frequency) of a received message tone is not necessarily centered in the selected time/frequency cell. Rather, it is randomly located in the cell, uniformly distributed in both time and frequency.

For a simple analysis of the magnitude of this effect, we consider the transmission of a Gaussian pulse. We assume it is received without distortion. Using matched filtering at the receiver, the output is equivalent to the sampling of a bivariate (both time and frequency) Gaussian function. When the receiver is perfectly synchronized, this function is sampled at its peak. In our case, we are sampling at a point randomly located in a time/frequency cell centered about the peak

of the Gaussian function. The expected value of this sampled point can be shown to be down (relative to the peak) by 2.05 dB. By halving the uncertainty in both time and frequency, as discussed earlier, this synchronization loss can be reduced to .61 dB.

The loss in signal energy due to spreading cannot be so readily estimated. It depends on both the form and degree of the spreading. Given a specific model for the spreading, the shape of the received waveform could be determined. This is done by convolving the pulse (time and frequency representations thereof) with the channel spreading functions. As the signal is stretched out (both in time and frequency), the peak value must decrease in order to maintain the original level of signal energy. This represents the spreading loss in the received signal.

Combining the two effects yields an estimate of the total required adjustment to the received SNR per tone to account for shifting and spreading losses. These tend to mollify each other---as the received pulse is spread out, the loss incurred by sampling off the peak decreases. Thus, the net effect will normally be less than that which would be obtained by considering the two effects separately and independently combining them.

SECTION 4

PERFORMANCE

In this section we discuss the statistics of the decision variable formed by the receiver under the hypotheses of no signal present and signal present. Two types of threshold, fixed and adaptive, against which the decision variable is to be compared, are discussed. Probabilities of false alarm and detection are defined for both cases. The mathematical formulations are briefly discussed and results are presented for the fixed threshold case.

As discussed earlier, an idealized situation is assumed. The channel, although fading, is non-dispersive; the effects of Doppler are known and removed at the receiver and the receiver is synchronized to the message. To relate these results to a particular actual example, we modify them as described in Section 3.2.3.

4.1 INTRODUCTION

Out of the received time/frequency matrix, N complex variables are selected according to the pre-defined tone pattern. Denote these as $\{u_i, 1 \le i \le N\}$. We consider two idealized hypotheses: 1) The $\{u_i\}$ are all purely noise; 2) The $\{u_i\}$ are all signal plus noise.

The noise terms are assumed to be independent, zero-mean, complex normal random variables of variance σ^2. We are considering only fading channels, hence, we assume the signal terms to also be independent, zero-mean, complex normal random variables of variance σ_s^2. As the individual tones are separated in time and frequency, the assumptions of independence are realistic. We then write the two hypotheses as:

1. $u_i = x_i + jy_i \quad ; \quad x_i, y_i \quad \varepsilon \quad N(0,\sigma^2)$
2. $u_i = x_i + jy_i \quad ; \quad x_i, y_i \quad \varepsilon \quad N(0,\sigma^2 + \sigma_s^2)$

where the notation "$x_i \varepsilon N(\mu,\sigma^2)$" is read as "$x_i$ is a normal random variable of mean μ and variance σ^2."

Detecting a signal under these conditions is a classic problem. The optimum receiver may be shown (Ref. 7) to be one in which a decision variable, formed by summing the squared magnitudes of the tone responses, is compared against a threshold determined by the variance of the noise. We denote this decision variable by U,

$$U = \sum_{i=1}^{N} x_i^2 + y_i^2$$

As the $\{x_i\}$ and the $\{y_i\}$ are identically distributed and mutually independent, we may equivalently write

$$U = \sum_{i=1}^{2N} x_i^2 \tag{4-1}$$

Note that U is essentially* a chi-squared random variable with 2N degrees of freedom.

The signal-to-noise ratio is defined by

$$\gamma = \frac{\sigma_s^2}{\sigma^2} \tag{4-2}$$

and

$$\alpha = \frac{1}{1+\gamma} = \frac{\sigma^2}{\sigma^2+\sigma_s^2} \tag{4-3}$$

The two hypotheses defining the distribution of the $\{x_i\}$ in equation (4-1) may then be written as

$$\left.\begin{array}{ll} x_i \; \varepsilon \; N(0,\sigma^2) & \text{, noise only} \\ x_i \; \varepsilon \; N(0,\sigma^2/\alpha) & \text{, signal} \end{array}\right\} \tag{4-4}$$

The test to determine whether or not an alert message is present consists of comparing the decision variable U against a threshold, which we denote as W. When U exceeds the threshold but a signal is not present, we have a false alarm; inversely, when U does not exceed the threshold but a signal is present, we have a missed detection. The probabilities for each of these forms of error are:

$$P_{FA} = P_r \{U \geq W \mid \text{no signal}\} \tag{4-5}$$

$$P_{MD} = P_r \{U < W \mid \text{signal}\} \tag{4-6}$$

The probability of a successful detection is

$$P_D = 1 - P_{MD} = P_r \{U \geq W \mid \text{signal}\} \tag{4-7}$$

The particular probability distributions to be used here depend upon the form of thresholding used, as described in the next two sections.

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*Strictly speaking, a chi-squared random variable is formed by summing the squares of unit variance normal random variables.

The traditional design approach in a detection problem is to first determine the maximum false-alarm rate at which one is willing to operate. Given this value of P_{FA}, equation (4-5) is inverted to determine the required threshold W. Using this threshold in equation (4-7), one then calculates the detection probability as a function of the received signal-to-noise ratio.

For this particular problem, the number of tones, N, is not specified. Thus, our objective is to create, for a given P_{FA}, a set of curves parametrized by N depicting detection probability as a function of signal-to-noise. Given a desired level of performance, described by the probability of detection realized at a certain signal-to-noise ratio and false-alarm rate, the appropriate number of message tones may then be determined. In actual practice, one would also have to determine the intercept probabilities corresponding to the chosen parameters in order to fully evaluate a particular signal design.

4.2 THE USE OF A FIXED THRESHOLD

4.2.1 Formulation

In this section we examine the use of a fixed value for the threshold W. This threshold is set to a multiple of σ^2, the variance of the noise, which implies having precise knowledge of this parameter. This is an idealization, yielding optimum performance. In the next section we will consider an adaptive form of thresholding. This effectively derives an estimate of σ^2 from measurements of background noise and represents a more realistic thresholding technique.

First we consider in greater detail the decision variable U under the hypothesis that no signal is present. From equation (4-1) we have

$$U = \sum_{i=1}^{2N} x_i^{\,2} \;,\quad x_i \in N(0,\sigma^2)$$

This may be converted to

$$t = \frac{U}{\sigma^2} = \sum_{i=1}^{2N} (x_i/\sigma)^2 \;,\quad (x_i/\sigma) \in N(0,1)$$

which is a chi-squared random variable with 2N degrees of freedom. The density function is

$$f_t(t) = \frac{t^{N-1}\, e^{-t/2}}{2^N\, \Gamma(N)}$$

where $\Gamma(N)$ is the gamma function

$$\left.\begin{aligned} \Gamma(N) &= \int_0^\infty t^{N-1}\, e^{-t}\, dt \\ \Gamma(N) &= (N-1)! \;,\; N \text{ an integer} \end{aligned}\right\} \qquad (4\text{-}8)$$

Alternatively, the decision variable U may be expressed in terms of the random variable v,

$$v = \frac{t}{2} = \frac{U}{2\sigma^2} \tag{4-9}$$

v is said to have a gamma distribution, its density is given by

$$f_v(v) = \frac{v^{N-1} e^{-v}}{\Gamma(N)} \tag{4-10}$$

We shall use this form of the decision statistic.

Next, we consider the decision variable under the alternative hypothesis (signal present). We now have

$$U = \sum_{i=1}^{2N} x_i^2 \;, \quad x_i \in N(0, \sigma^2/\alpha)$$

As before, we form $v = U/2\sigma^2$. The presence of a signal necessitates an additional scale factor to convert the decision variable to one expressed in terms of unit variance normals. Hence,

$$v = \frac{1}{\alpha} \left[\frac{1}{2} \sum (\sqrt{\alpha}\; x_i/\sigma)^2 \right] = \frac{1}{\alpha} v' \tag{4-11}$$

The density of v is trivially related to that of v', further, the density of v' is the same as that for v under the noise only hypothesis as given by equation (4-10)

Thus,

$$f_v(v) \Big|_{\text{signal},\alpha} = \alpha f_v(\alpha v) \Big|_{\text{noise only}} \tag{4-12}$$

defines the probability density function of the statistic v when there is a signal present.

The ideal fixed threshold W is a multiple of the noise variance,

$$W = 2\Lambda\sigma^2 \tag{4-13}$$

Comparison of U against W is then equivalent to the test

$$v \overset{?}{\geq} \Lambda \tag{4-14}$$

with v defined by equations (4-9) and (4-11). We will use this formulation and refer for simplicity to Λ itself as the threshold.

Inserting into equations (4-5) and (4-7), we get

$$P_{FA} = \int_{\Lambda}^{\infty} f_v(v)\, dv \Bigg|_{\text{noise}}$$

$$P_D = \int_{\Lambda}^{\infty} f_v(v)\, dv \Bigg|_{\text{signal}}$$

By (4-12), the latter may also be expressed in terms of the density for the noise-only hypothesis. Dropping the designator, the expressions we need to evaluate become

$$P_{FA} = \int_{\Lambda}^{\infty} f_v(v)\, dv \tag{4-15}$$

$$P_D = \int_{\alpha\Lambda}^{\infty} f_v(v)\, dv \tag{4-16}$$

with $f_v(v)$ defined in equation (4-10).

Although the formulations presented for P_{FA} and P_D are conceptually identical, our uses of them are not. Equation (4-15) must be inverted, i.e., given P_{FA}, one wishes to solve for Λ. Then, this value is used directly in equation (4-16) to evaluate P_D.

We first discuss direct evaluation of the integrals. This is described in terms of P_{FA}, although it also applies to P_D. Then, inversion of the integral is considered.

4.2.2 Solution

Two approaches are available for integration of the density given in equation (4-10). It may be evaluated as a series consisting of N terms. Specifically, expressing $\Gamma(N)$ as $(N-1)!$, it is trivial to show that

$$\int e^{-v} \frac{v^{N-1}}{(N-1)!} dv = -e^{-v} \sum_{i=0}^{N-1} \frac{v^i}{i!} \qquad (4\text{-}17)$$

Although the above is an exact form, it is often too cumbersome for evaluation. Examination of it, however, yields some useful insight. The summation consists of the first N terms of the series representation of e^v; it is often denoted as $e_{N-1}(v)$. In this form, we have

$$P_{F\Lambda} = e^{-\Lambda} e_{N-1}(\Lambda) \qquad (4\text{-}18)$$

Examining equation (4-18), we see that for a fixed Λ, as N goes to infinity, P_{FA} approaches unity. In retrospect, this is not surprising, as, in order to maintain a given P_{FA}, we expect Λ to grow with N. Actually, it is mathematically more meaningful (see Appendix A) to express Λ in terms of (N-1). We will define a "normalized" threshold β by

$$\beta = \frac{\Lambda}{N-1} \qquad (4\text{-}19)$$

Then, we may write

$$P_{FA} = \frac{e_{N-1}(\beta(N-1))}{\exp(\beta(N-1))}$$

From equation (6.5.34) of Ref. 4,

$$\lim_{(N-1)\to\infty} P_{FA} = \begin{cases} 0, & \beta>1 \\ 1/2, & \beta=1 \\ 1, & 0\le\beta<1 \end{cases} \qquad (4\text{-}20)$$

Calculation of values of Λ shows that $\beta>1$. Hence, as the message length grows infinitely large, the false-alarm probability becomes vanishingly small. Thus, we can expect our system performance to improve as N increases.

For small values of N, evaluation of equation (4-18) is feasible. As N (and consequently Λ) grows large, the magnitudes of the terms become increasingly difficult to handle. As an example, values of N = 32 and Λ = 66.39 yield (Ref. 5) a false-alarm probability of 10^{-6}. Use of equation (4-18) would require first forming the terms exp $(-\Lambda) \approx 10^{-29}$; $e_{N-1}(\Lambda) \approx 10^{+23}$ in order to conclude that P_{FA} was 10^{-6}. N = 32 is hardly to be considered large, in fact, we are interested in values such as N = 100. Then, the exponents involved in (4-18) are such that maintaining numerical accuracy can become a serious problem. We further note that inversions of (4-18) is clearly extremely awkward.

Returning to equation (4-15), we present an alternative solution. Equation (4-8) defines the (complete) gamma function. Similarly, one has the incomplete gamma function and its complement;

$$\left.\begin{aligned} \gamma(N,x) &\triangleq \int_0^x t^{N-1} e^{-t} \, dt \\ \Gamma(N,x) &\triangleq \int_x^\infty t^{N-1} e^{-t} \, dt \\ \gamma(N,x) + \Gamma(N,x) &= \Gamma(N) \end{aligned}\right\} \qquad (4\text{-}21)$$

Then, one may write

$$P_{FA} = \frac{\Gamma(N,\Lambda)}{\Gamma(N)} \qquad (4\text{-}22)$$

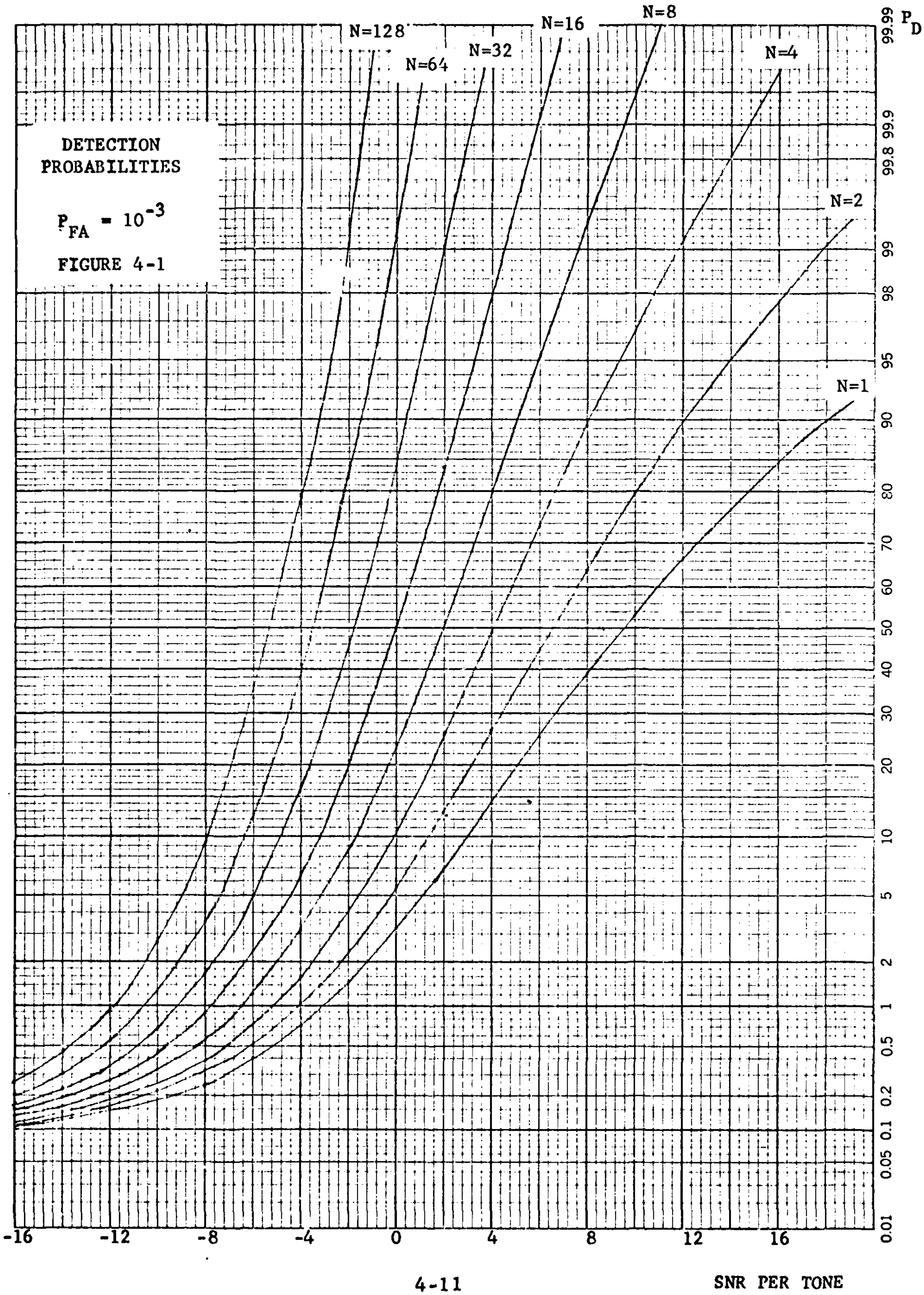
which is known as an incomplete gamma ratio.

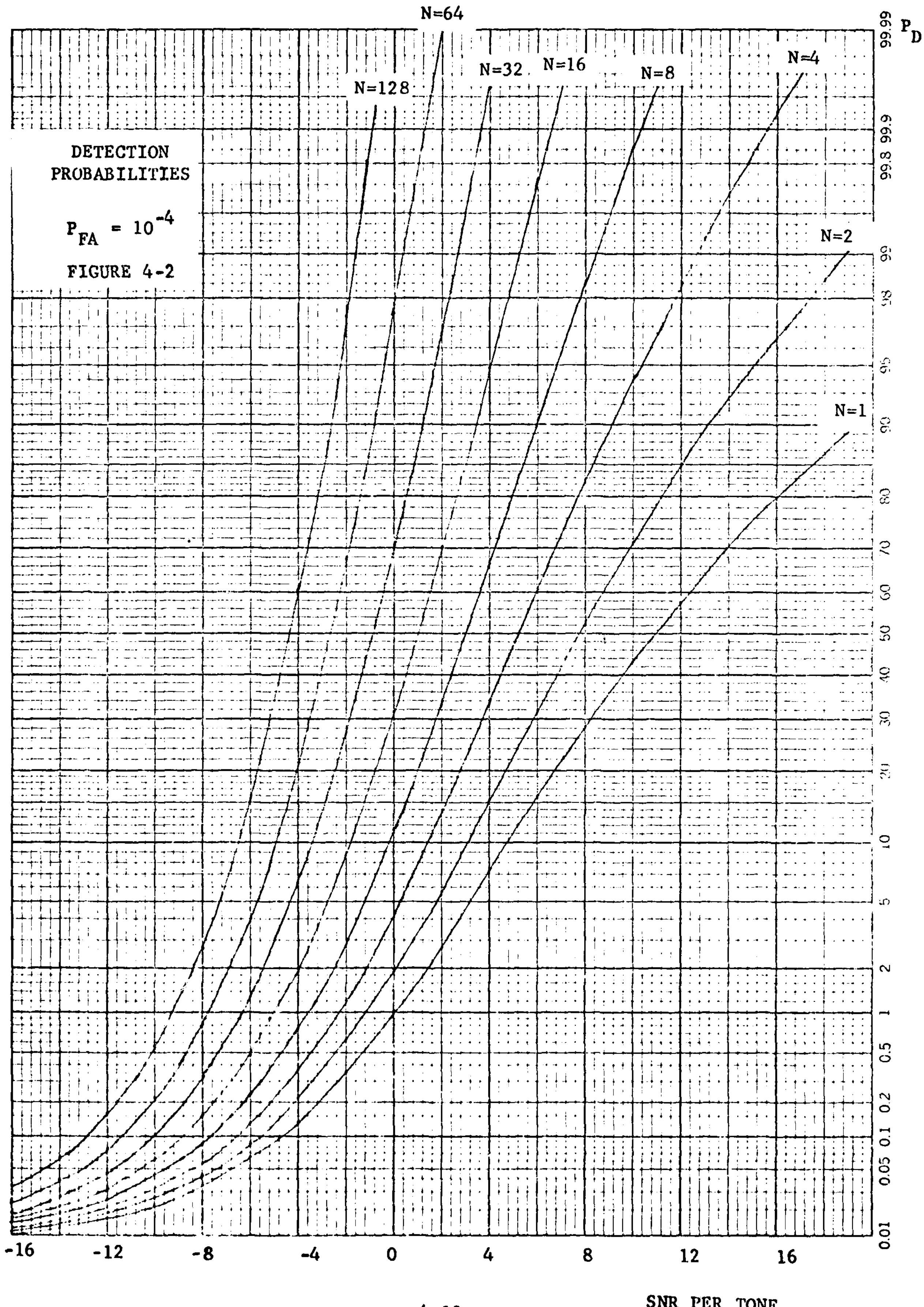
The gamma functions, complete and incomplete, have been rather thoroughly studied, and many approximations or expansions are tabulated (e.g., Refs. 4 and 6). Surprisingly, much less work appears to have been done on the incomplete gamma ratio. In Appendix A, we present what we consider to be some new results on this problem. An approximate solution is derived, valid for calculation of the false-alarm probability, which is readily invertable for determination of the threshold Λ. The accuracy increases as N grows large or as the desired P_{FA} grows small, which are the two conditions that aggravate use of the direct summation approach.

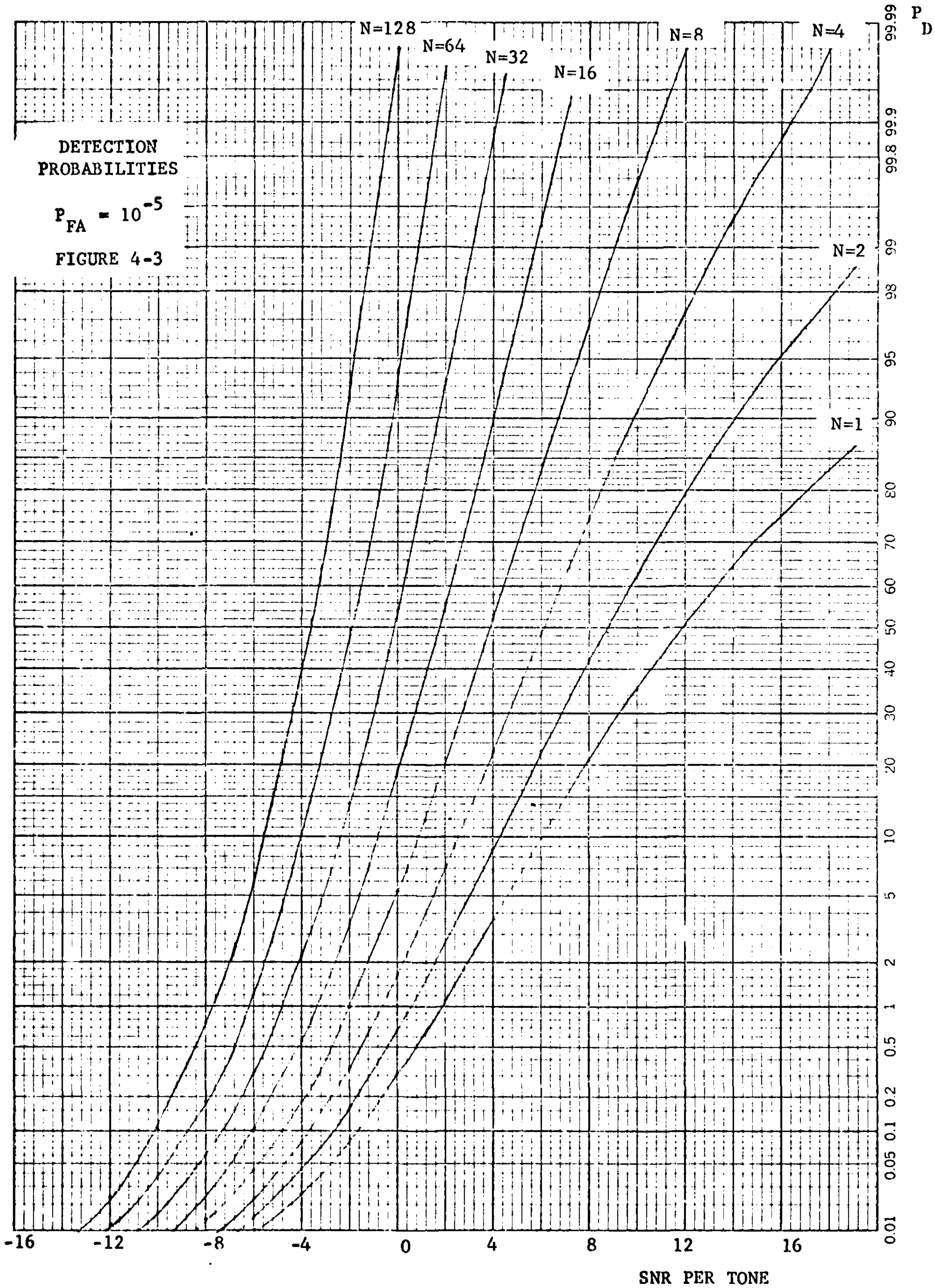
This approach unfortunately does not hold for calculation of the detection probability, except for very small signal-to-noise ratios, as the asymptotic expansion upon which it is based is invalid for values of the argument $(\alpha\Lambda)$ that appear when one writes equation (4-16) in the form of an incomplete gamma ratio. Thus, although the more difficult problem of inversion of the false-alarm integral is solved rather easily, calculation of the detection probability integral must be done in a manner similar to that given in (4-18). Actually, a modified form of this, given in Ref. 5, is used. This is also explained in Appendix A.

4.2.3 Results

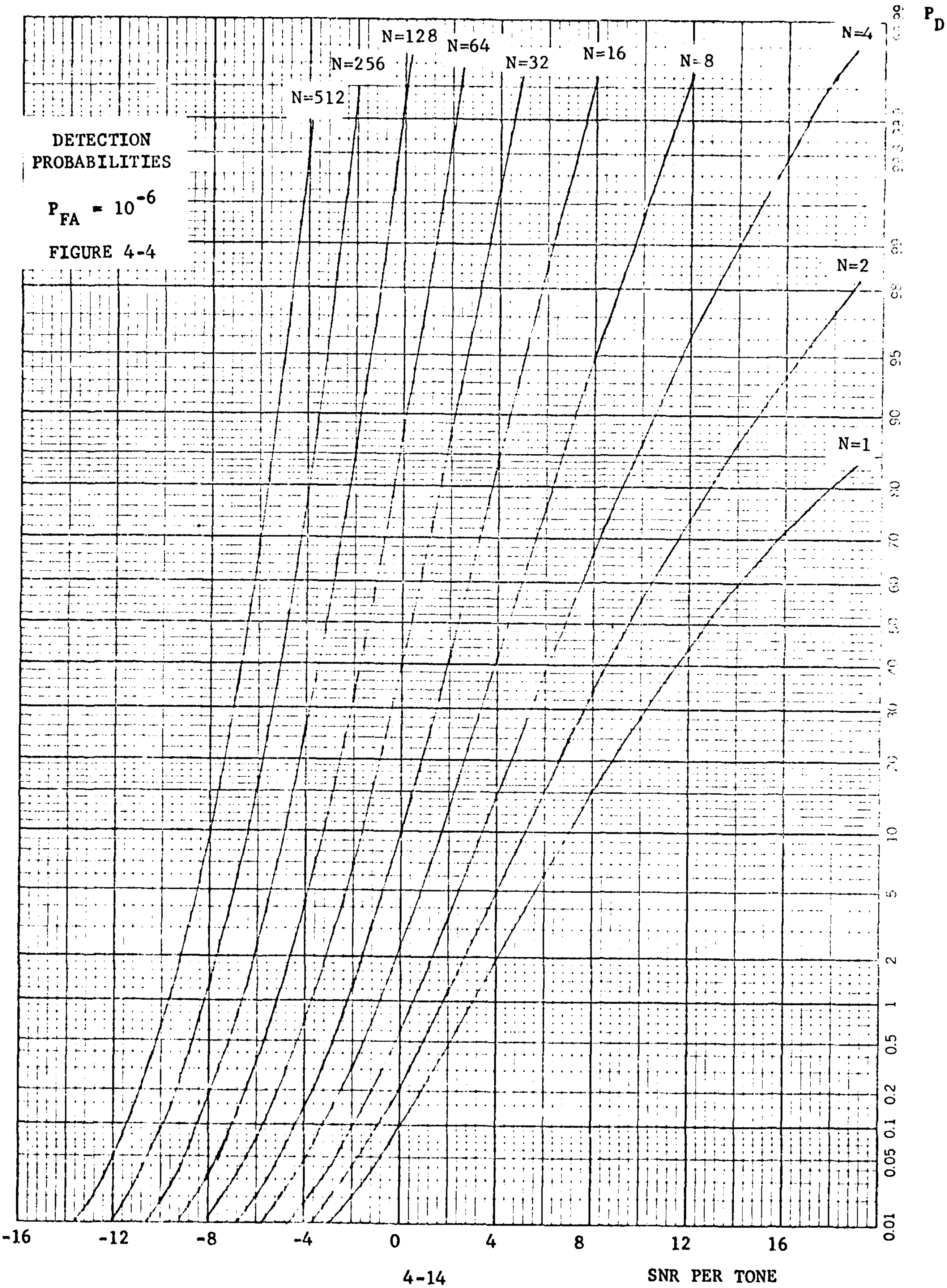
In the next few pages, we present families of performance curves (Figures 4-1 through 4-7) for a system using a fixed thresholding technique. The curves are presented first by the operating level at which the false-alarm probability is set and then by different values of N, the message length. P_D is the detection probability in % and E/n_o is the signal-to-noise ratio per tone in dB.


DETECTION
PROBABILITIES
$P_{FA} = 10^{-3}$
FIGURE 4-1
N=128
N=64
N=32
N=16
N=8
N=4
N=2
N=1
P_D
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
-16
-12
-8
-4
0
4
8
12
16
SNR PER TONE


DETECTION PROBABILITIES
$P_{FA} = 10^{-4}$
FIGURE 4-2
N=128
N=64
N=32
N=16
N=8
N=4
N=2
N=1
P_D
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
-16
-12
-8
-4
0
4
8
12
16
SNR PER TONE



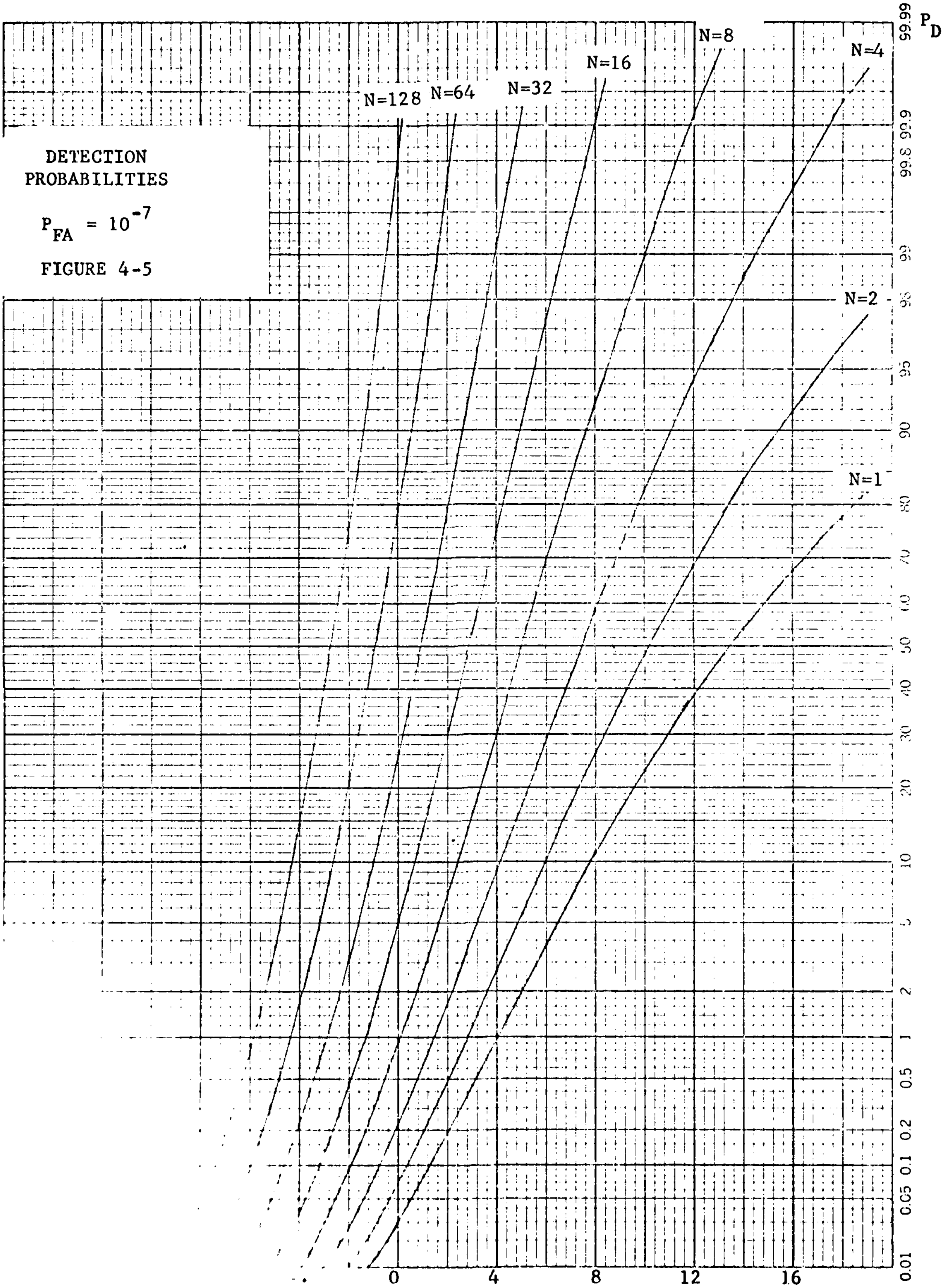
DETECTION
PROBABILITIES
$P_{FA} = 10^{-5}$
FIGURE 4-3
N=128
N=64
N=32
N=16
N=8
N=4
N=2
N=1
P_D
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
-16
-12
-8
-4
0
4
8
12
16
SNR PER TONE



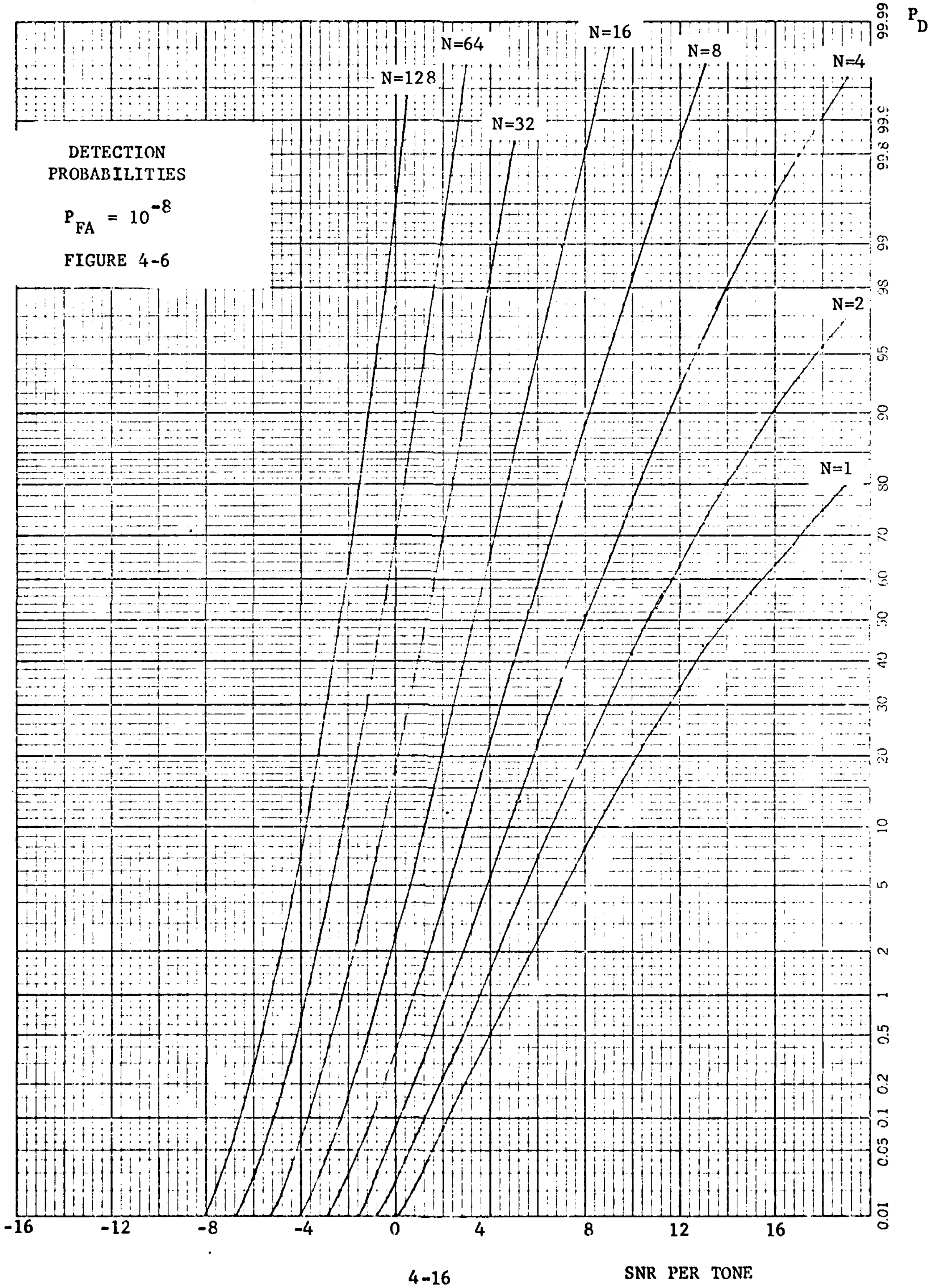
DETECTION PROBABILITIES

$P_{FA} = 10^{-6}$

FIGURE 4-4



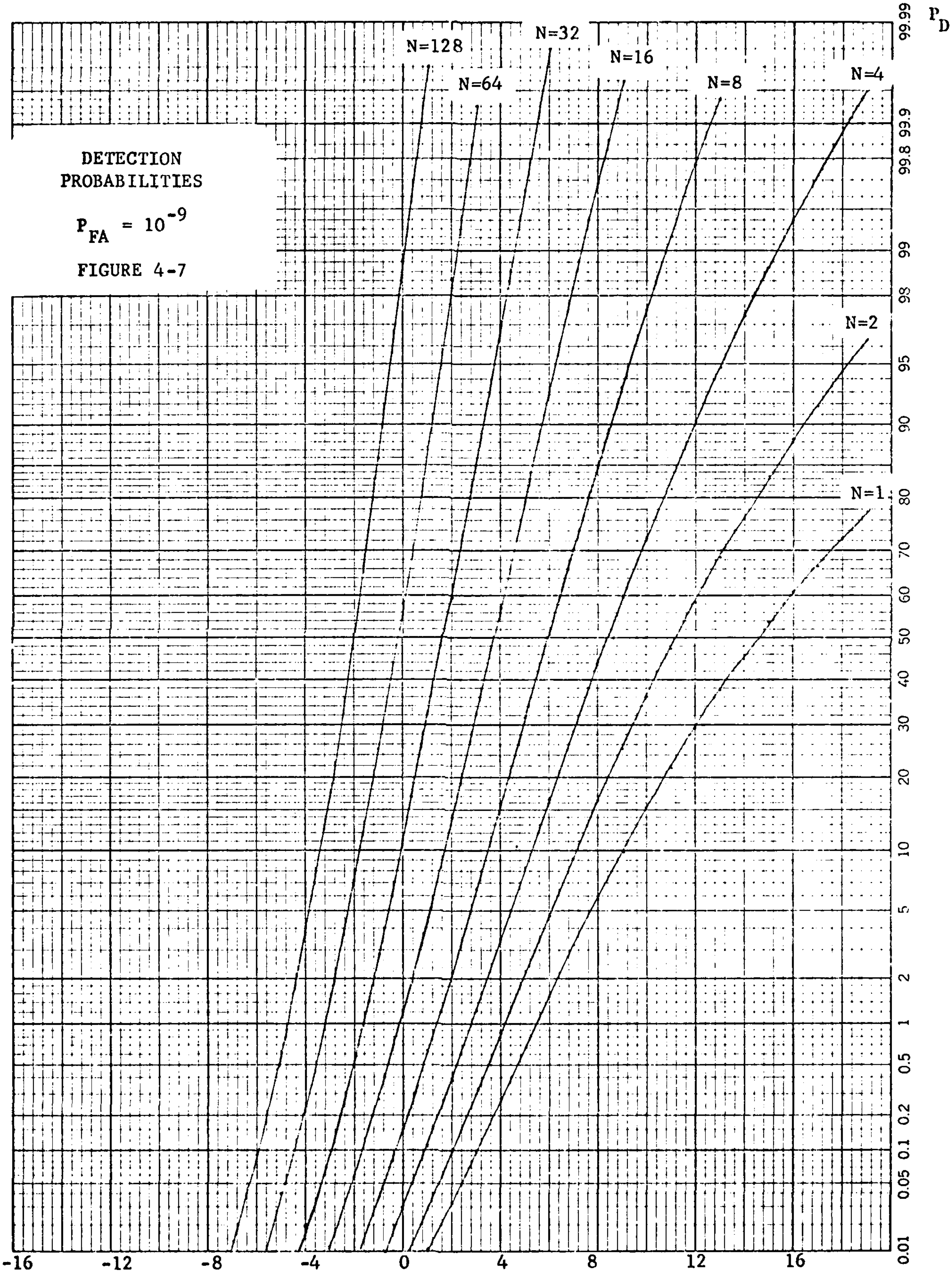
DETECTION
PROBABILITIES
$P_{FA} = 10^{-7}$
FIGURE 4-5
N=128
N=64
N=32
N=16
N=8
N=4
N=2
N=1
P_D
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
0
4
8
12
16
SNR PER TONE



DETECTION PROBABILITIES

$P_{FA} = 10^{-8}$

FIGURE 4-6


DETECTION
PROBABILITIES
$P_{FA} = 10^{-9}$
FIGURE 4-7
N=128
N=64
N=32
N=16
N=8
N=4
N=2
N=1
P_D
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
-16
-12
-8
-4
0
4
8
12
16
SNR PER TONE

A simple interpretation of P_{FA} is given in Table 4-1. Here, the false-alarm interval (denoted τ_{FA}) is defined as the length of time over which the probability of at least one false-alarm occurring is one-half. The values of τ_{FA} given are normalized to a tone of length T = 1 ms. (Note that although the overall message length is NT, the receiver described in Section 3.1 makes a decision every T seconds.) Thus, for instance, if the tone length is 20 ms and the operating level of the system is set for $P_{FA} = 10^{-6}$, a false-alarm will occur, on the average, every 3.8 hours.

TABLE 4-1

FALSE ALARM INTERVALS

τ_{FA} = False Alarm Interval, normalized for T = 1 ms

$P_{FA} = 10^{-P}$

P	τ_{FA}
3	.69 seconds
4	6.9
5	1.15 minutes
6	11.5
7	1.91 hours
8	19.1
9	7.99 days

4.3 THE USE OF A VARIABLE THRESHOLD

4.3.1 Formulation

In this section, we examine the use of a variable or adaptive thresholding technique. Just as the decision variable U is formed by summing the squared magnitudes of N tones which may or may not contain a signal, so too is the threshold W formed by summing the squared magnitudes of N tones believed to be purely noise. These tones may be derived from a frequency band wherein no signal is ever expected, alternatively, they may be from past decisions which were determined to represent the noise-only hypothesis. Actually, one would create the noise reference from more than only N samples in order to obtain a better estimate of the variance of the noise process. Such a more general form is presented, the implications of which are discussed later.

The statistic U is as before,

$$U = \sum_{i=1}^{2N} x_i^2 \quad , \quad \begin{cases} x_i \; \varepsilon \; N(0,\sigma^2) & \text{, noise} \\ x_i \; \varepsilon \; N(0,\sigma^2/\alpha) & \text{, signal} \end{cases}$$

The threshold W is given by the average over ν intervals of length N,

$$W = \frac{1}{\nu} \sum_{i=1}^{\nu 2N} w_i^2 \tag{4-23}$$

where, for simplicity, ν is assumed integral. The test consists of comparing U to some multiple of W, specifically

$$\left. \begin{aligned} & U \overset{?}{\gtrless} \beta W \\ \text{or} \qquad & F \triangleq \frac{U}{W} \overset{?}{\gtrless} \beta \end{aligned} \right\} \tag{4-24}$$

Using this form, we will refer to the scalar β as the threshold and F as the decision variable.

The decision variable may be expressed as

$$F = \frac{\frac{1}{2N} \quad (U/\sigma^2)}{\frac{1}{2N} \quad (W/\sigma^2)} = \frac{\frac{1}{2N} \sum_{i=1}^{2N} (x_i/\sigma)^2}{\frac{1}{\nu 2N} \sum_{i=1}^{\nu 2N} (w_i/\sigma)^2}$$

Under the noise only hypothesis, this is the ratio of two chi-squared variates, each normalized by its number of degrees of freedom. Such a random variable is known as an F-distribution* with 2N and $\nu 2N$ degrees of freedom. Its density function is given by

$$f_F(v) = \frac{(2N)^N (\nu 2N)^{\nu N}}{B(N,\nu N)} \cdot \frac{v^{N-1}}{(\nu 2N + 2Nv)^{(\nu+1)N}}$$

where B (m,n) is the (complete) beta function defined by

$$B(m,n) = \int_0^1 t^{m-1} (1-t)^{n-1} dt \tag{4-25}$$

We thus have

$$P_{FA} = \int_\beta^\infty f_F(v) \, dv \tag{4-26}$$

We derive the detection probability in a similar manner. The presence of the signal necessitates a different normalization, viz.

$$F = \frac{\frac{1}{2N} \frac{1}{\alpha} (U\alpha/\sigma^2)}{\frac{1}{2N} (W/\sigma^2)} = \frac{1}{\alpha} F'$$

The density for F' is the same as that for F under the noise only hypothesis. As was the case for a fixed threshold, P_D has a formulation identical to P_{FA},

$$P_D = \int_{\alpha\beta}^\infty f_F (v) \, dv \tag{4-27}$$

*More specifically, a doubly central F to denote that each of the chi-squareds are central (formed from normal r.v.s. of zero-mean).

4.3.2 Solution

It can be shown (Ref. 3c) that an F-distribution can be expressed as a beta distribution. Let F be a random variable having an F-distribution with 2N and ν2N degrees of freedom; let t be a random variable having a beta distribution with parameters N and νN. Then,

$$Pr\left[F \le K\right] = Pr\left[t \le K'\right] \tag{4-28}$$

where

$$K' = \frac{K}{\nu + K} \tag{4-29}$$

The probability that $t \le K'$ is given by an incomplete beta ratio, i.e.,

$$\left.\begin{aligned} Pr\left[t \le K'\right] &= I_{K'}(N, \nu N) \\ \text{where} \quad I_x(m,n) &= \frac{B_x(m,n)}{B(m,n)} \end{aligned}\right\} \tag{4-30}$$

and B_x is the incomplete beta function

$$B_x(m,n) = \int_0^x t^{m-1}(1-t)^{n-1}dt \tag{4-31}$$

The incomplete beta ratio has a useful symmetry property

$$I_x(m,n) = 1 - I_{1-x}(n,m) \tag{4-32}$$

Using these identities, we may then write

$$\left.\begin{aligned} P_{FA} &= Pr\left[F \ge \beta\right] = 1 - Pr\left[F \le \beta\right] \\ P_{FA} &= 1 - I_{\beta'}(N, \nu N) \\ &= I_{\beta''}(\nu N, N) \\ \text{with } \beta' &= \frac{\beta}{\nu+\beta}, \quad \beta'' = \frac{\nu}{\nu+\beta} = 1 - \beta' \end{aligned}\right\} \tag{4-33}$$

and similarly

$$\left.\begin{aligned} P_D &= Pr\left[F \ge \alpha\beta\right] \\ &= 1 - I_{(\alpha\beta)'}(N, \nu N) = I_{(\alpha\beta)''}(\nu N, N) \\ \text{with} \quad (\alpha\beta)' &= \frac{\alpha\beta}{\nu+\alpha\beta}, \quad (\alpha\beta)'' = \frac{\nu}{\nu+\alpha\beta} = 1 - (\alpha\beta)' \\ \text{and} \quad (\alpha\beta)' &= \frac{\alpha\,\beta'}{(1-\beta')+\alpha\,\beta'}, \quad (\alpha\beta)'' = \frac{\beta''}{\beta''+\alpha(1-\beta'')} \end{aligned}\right\} \tag{4-34}$$

The formulations involved for analyzing performance using a variable threshold are rather similar to those for a fixed threshold, except that they involve beta rather than gamma functions. Unfortunately, much less investigation into the properties of the beta functions appears to have been done---at least, no relations resulting in a feasible inversion of (4-33) have been found.

One could use expression (4-31) for the incomplete beta function, expand the integrand into n terms and integrate each directly. This is equivalent to summing n binomial probabilities. From this, the incomplete beta ratio could be evaluated. As shown in the next section, the computational effort involved here does not appear to be justified.

4.3.3 Discussion

It is well known that, as the variable threshold is "noisier" than an ideal fixed one, such a system would have to operate at a higher threshold setting in order to operate at the same false-alarm rate as a fixed system. Thus, one would expect a lower detection probability and consequently, poorer performance from a variable system. (This, of course, supposes that the fixed thresholding technique always has perfect knowledge of the variance of the noise process.) Further, we suspect that forming the threshold by averaging over a greater number of observations will provide a smoother estimate of the noise and hence a performance approaching that of the fixed system. It is easy to verify these intuitions.

We earlier discussed the fact that forming the threshold over νN observations and normalizing by $1/\nu$ allowed a greater history of the

noise process to be used. We now examine the behavior of such a threshold as ν grows large. As defined in equation (4-23), W may be written as

$$W = \frac{\sigma^2}{\nu} \quad \chi^2_{\nu 2N}$$

i.e., a chi-squared with (ν2N) degrees of freedome multiplies by the scaling factor (σ^2/ν). Since, in general,

$$E\left[\chi^2_N\right] = N$$

$$\mathrm{var}\left[\chi^2_N\right] = 2N$$

we have

$$E\left[W\right] = 2N\sigma^2$$

$$\mathrm{var}\left[W\right] = \frac{2N\sigma^4}{\nu}$$

Letting ν approach infinity,

$$\left.\begin{aligned} \lim_{\nu\to\infty} \; & E\left[W\right] = 2N\sigma^2 \\ \lim_{\nu\to\infty} \; & \mathrm{var}\left[W\right] = 0 \end{aligned}\right\} \qquad (4\text{-}35)$$

Thus, as the sample size is increased, the "variable" threshold tends towards a fixed threshold as discussed in Section 4.2. Specifically, note that the test defined in equation (4-24) then becomes

$$U \overset{?}{\geq} 2N\beta\sigma^2 \quad .$$

which, other than a minor difference in notation, is identical to that presented for the fixed thresholding technique.

We can also examine the system performance as $\nu\to\infty$. For simplicity, we fix N = 1. Then, $I_x(\nu,1) = x^\nu$ and the equation for P_{FA}, in this simple case, is readily inverted for β''. The quantity $(\alpha\beta)''$ may be expressed in terms of this via (4-34), thereby allowing determination of P_D. For large ν,β'' is close to unity, specifically,

$$\beta'' = 1 - \varepsilon \quad ; \quad \varepsilon = \frac{-\ln P_{FA}}{\nu}$$

As $\nu \to \infty$, it is then easy to show that

$$P_D = (P_{FA})^{\alpha} = (P_{FA})^{\frac{1}{1+\gamma}} \qquad (4\text{-}36)$$

which is precisely the result one would get for a fixed threshold analysis. This same approach can be used to show inferior performance for finite values of ν.

We therefore conclude that although the variable thresholding method will produce performance inferior to a fixed threshold for small ν, in actual practice, this situation will not occur. Neither will an ideal fixed threshold occur. Rather, an implementable system will have an adaptive thresholding scheme with a sufficiently large choice of ν to render it equivalent to a fixed system, with its performance then as given in Section 4.2.3.

SECTION 5

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APPENDIX A

APPROXIMATIONS TO THE INCOMPLETE GAMMA RATIO

The incomplete gamma ratio is defined as the ratio of the complementary incomplete gamma function to the complete. No nomenclature is established for this quantity; we shall denote the function by $P(N,\Lambda)$, i.e.,

$$P(N,\Lambda) = \frac{\Gamma(N,\Lambda)}{\Gamma(N)} \tag{A-1}$$

and a particular evaluation thereof by simply P.

We consider here the solution of two related problems. The first is the direct-given N and Λ, we wish to find P. The second is an inversion---given P and N, we wish to find Λ. Approximations for these two problems, for certain ranges of values of N and Λ, will be obtained appropriate to the discussions in Section 4.2.2.

We first define a normalized threshold,

$$\beta = \frac{\Lambda}{N-1} \tag{A-2}$$

The choice of (N-1) rather N as a normalizing factor is not logically obvious, but is dictated rather by the formulations which follow. We may then write the ratio as $P(N,\beta)$.

The first approximation we derive is valid only for $\beta > 1$. This is of interest in the calculation of false-alarm probabilities and has the important advantage of being readily inverted. We approximate $P(N,\Lambda)$ by forming the quotient of asymptotic expansions for each of the functions appearing in (A-1). Although these expansions are separately known, we have not seen this combination nor the useful result it produces.

From equation 9.5.4 of Ref. 6, we have the asymptotic expansion

$$\Gamma(N,\Lambda) \approx \frac{e^{-\Lambda}\,\Lambda^{N}}{[\Lambda-(N-1)]}\left[1-\frac{(N-1)}{[\Lambda-(N-1)]^{2}}+\frac{2(N-1)}{[\Lambda-(N-1)]^{3}}+0\left[\frac{(N-1)^{2}}{[\Lambda-(N-1)]^{4}}\right]\right] \quad \text{(A-3)}$$

This is valid as N and Λ each grow large provided that the quantity $z \triangleq \frac{\sqrt{N-1}}{\Lambda-(N-1)}$ goes to zero from the positive side. From equation (A-2), we have

$$[\Lambda-(N-1)] = (N-1)(\beta-1) \quad \text{(A-4)}$$

Then, $z = \frac{1}{(\beta-1)\sqrt{N-1}}$ goes to zero provided that β does not go to unity. In actuality, β does approach unity and for values of $\beta \simeq 1$ the approximation we derive will be incorrect. However, for values of P and N of interest to us, (A-3) is seen to be sufficiently accurate. Note that if $\Lambda<(N-1)$, as would be the case in evaluation of detection probabilities, the parameter z is negative and the asymptotic form is invalid. Equation (A-3) may be re-written as

$$\Gamma(N,\Lambda) \approx \frac{e^{-\Lambda}\,\Lambda^{N}}{(N-1)(\beta-1)}\left[1-\frac{1}{(N-1)(\beta-1)^{2}}+\frac{2}{(N-1)^{2}(\beta-1)^{3}}+0\left[\frac{1}{(N-1)^{2}(\beta-1)^{4}}\right]\right] \quad \text{(A-5)}$$

For brevity, we will express the term in brackets as [1 - etc.]. We will assume some if not all of the "etc." terms negligible in the following analysis. Clearly, if $\beta \rightarrow 1$, the terms we ignore become the dominant ones.

The expansion we use for the denominator of (A-1) is the familiar Stirling approximation:

$$\Gamma(N) = \sqrt{2\pi}\,(N-1)^{(N-1/2)}\,e^{-(N-1)}\,(1+\text{small}) \quad \text{(A-6)}$$

The terms designated "small" are tabulated, we will maintain only the leading term (even this could be rather safely ignored) which is 1/12(N-1).

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Combining equations (A-5) and (A-6), we now have

$$P(N,\Lambda) \approx \frac{e^{-\Lambda}}{e^{-(N-1)}} \times \frac{\Lambda^N}{(N-1)^{(N-1/2)}} \times \frac{1}{\sqrt{2\pi}} \times \frac{1}{(N-1)(\beta-1)} \times \frac{[1 - \text{etc.}]}{(1 + \text{small})}$$

or

$$P(N,\beta) \approx \frac{e^{-(N-1)(\beta-1)} \quad \beta^N \sqrt{N}}{\sqrt{2\pi} \ (N-1)(\beta-1)} \quad (\sim 1)$$

where "($\sim$1)" denotes the ratio of the last two terms. This is further simplified by the substitution

$$\beta^N = \beta \ e^{(N-1) \ln \beta}$$

Then,

$$P(N,\beta) \approx \frac{1}{\sqrt{2\pi}} \ \frac{\sqrt{N}}{(N-1)} \ \frac{\beta}{(\beta-1)} \ e^{-(N-1) \ [(\beta-1)-\ln \beta]} \quad (\sim 1) \qquad \text{(A-7)}$$

When N and β are known, this may be readily solved for P. As many terms as are deemed appropriate may be maintained in ($\sim$1).

To invert equation (A-7), we take natural logarithms and solve for β.

$$\ln P = -(N-1) \ [(\beta-1)-\ln\beta] + \ln\beta - \ln(\beta-1)$$
$$+ 1/2 \ln N - \ln (N-1) - 1/2 \ln (2\pi) + \ln (\sim 1) \qquad \text{(A-8)}$$

The term ($\sim$1) depends upon β; to get an initial estimate, we ignore it. To simplify, we express the above equation in terms of only β or (β-1). That is, we utilize one of the following approximations:

$$\ln (\beta-1) \approx \beta-2, \ \beta \approx 2$$

or

$$\ln (\beta) \approx \beta-1, \ \beta \approx 1$$

For $\beta = 1.58$, these are equally in error. This maximum error is equal to +.12, equivalent to a multiplicative error in P of 1.12.

We now have two possible cases, specifically,

$$\left.\begin{array}{l} \ln(\beta-1) \approx \beta-2, \quad \beta > 1.58 \\ \text{or} \\ \ln\beta \approx \beta-1, \quad \beta < 1.58 \end{array}\right\} \qquad \text{(A-9)}$$

of course, equation (A-9) implies a rough knowledge of the variable for which we wish to solve. This is often available, especially in a tabulation approach, wherein P might be held constant and β calculated for increasing values of N. Then, previous answers supply the necessary estimate. When no estimate is available, one may choose at random or try both methods and compare. Depending upon the equation chosen for (A-9), one of the following results.

Case 1. $\beta > 1.58$

Equation (A-8) is expressed entirely in terms of β. Define (for fixed P and N, this is a constant)

$$K = \frac{N - \ln P + 1/2\ \ln N - \ln(N-1) + (1 - 1/2\ \ln 2\pi)}{N} \qquad \text{(A-10)}$$

Then, we want to solve

$$\beta - \ln\beta = K \qquad \text{(A-11)}$$

Case 2. $\beta < 1.58$

Let $\beta' = (\beta-1)$. Define a constant

$$K' = -\ln P + 1/2\ \ln N - \ln(N-1) - 1/2\ \ln 2\pi = N(K-1)-1 \qquad \text{(A-12)}$$

We then wish to solve

$$\ln\beta' - \beta' = K'$$

In either case, we have a simple transcendental equation which does not admit of direct solution. However, an iterative technique (such as Newton's) is readily applied.

$$\begin{array}{ll} \text{Defining} & f(\beta) = \beta - \ln\beta - K, \\ & f'(\beta) = 1 - 1/\beta \end{array} \qquad \text{(A-14)}$$

Newton's method for solving (A-11) is:

$$\beta_{i+1} = \beta_i - \frac{f(\beta_i)}{f'(\beta_i)} \qquad \text{(A-15)}$$

The initial estimate is obtained from

$$\ln\beta \approx (\beta-1) - 1/2\ (\beta-1)^2$$

whence

$$\beta_o = 1 + \sqrt{2(K-1)} \qquad \text{(A-16)}$$

A similar method holds for the solution of (A-13).

Solution of either (A-11) or (A-13) yields an approximate inversion of the incomplete gamma ratio. Dependent upon the magnitudes of P and N and the desired accuracy of Λ, the result may be sufficient. If the term ($\sim$1), neglected in the solution above, is considered significant, the calculated value of β may be used as an initial estimate in a more exact iteration using equation (A-8). Alternatively, it may be used as the initial estimate in the other method we describe.

We present here a few examples illustrating the accuracy of the preceeding method. The validity of the results was tested by comparison against a table of solutions to the inverse problem given in Ref. . In general, the larger N or the smaller P, the more accurate the approximation.

1. from the tables, $N = 100$, $\Lambda = 187.248 \Rightarrow P = 10^{-12}$.
   Using the direct solution, as in equation (A-7), yields
   $P = 1.005 \times 10^{-12}$.

2. Inverting the same problem, using equations (A-10), (A-11), and (A-14) thru (A-16), yields $\Lambda = 187.272$. The error either here or above is presumably insignificant.

3. A similar inversion, but for smaller numbers. From the tables, $N = 32$, $P = 10^{-6} \Rightarrow \Lambda = 66.3937$. The calculated root is $\Lambda = 66.4513$.

4. Letting N get smaller, and P larger, we begin to notice errors which can be deemed significant.

   a) $N = 10$, $P = 10^{-1} \Rightarrow \Lambda = 14.206$. The calculated value is $\Lambda = 14.490$.

   b) Using $N = 10$ and the calculated Λ, the direct evaluation yields $P = .83 \times 10^{-1}$. Using Λ from the tables, $P = .92 \times 10^{-1}$.

These examples point out an interesting behavior. Although the inverse solution is derived from the direct one by further approximations and simplifications, it produces more accurate results. This is simply a matter of relative sensitivities. By examining the tables in Ref. 5., a small change in Λ (a few percent) is seen to be sufficient to change P by an order of magnitude.

5. The last example uses the smallest possible N and largest P and should represent worst performance for the approximation. Of course, in this case, the problem could be solved without recourse to approximation as explained in Section 4.2.2. Nonetheless, let $N = 2$ and $P = 10^{-1}$. The true solution is $\Lambda = 3.8897$; the calculated inverse is $\Lambda = 3.6636$. Evaluating directly with the true root yields $P = 1.58 \times 10^{-1}$.

We discuss now an alternative computational method for evaluating and/or inverting the incomplete gamma ratio. This is the method described by Pachares in Ref. 5. We use it here primarily to calculate detection

probabilities, where $\Lambda < (N-1)$, and the previous approximation breaks down. Pachares used it to invert the false-alarm equation to calculate the values of Λ in his tables. To ensure accuracy in determination of the performance results given in Section 4.2.3, we also do this. However, the value of Λ given by the preceeding method is used as an initial estimate in Pachares' iterative approach. This is more convenient to use than the initial estimate he considers.

As discussed in Section 4.2.2, $P(N,\Lambda)$ may be evaluated exactly as a sum of N terms,

$$P(N,\Lambda) = e^{-\Lambda} \sum_{k=0}^{N-1} \frac{\Lambda^k}{k!}$$

This may be re-structured to simplify the computations.

Define
$$S(N,\Lambda) = \frac{(N-1)!}{\Lambda^{N-1}} \sum_{k=0}^{N-1} \frac{\Lambda^k}{k!} \qquad \text{(A-17)}$$

Then,
$$P(N,\Lambda) = \frac{e^{-\Lambda}\,\Lambda^{N-1}}{(N-1)!}\; S(N,\Lambda) \qquad \text{(A-18)}$$

By suitable manipulation, $S(N,\Lambda)$ can be expressed in a recurrence relationship,

$$\left.\begin{aligned} S(N,\Lambda) &= 1 + \frac{N-1}{\Lambda}\, S(N-1,\Lambda) \\ S(1,\Lambda) &= 1 \end{aligned}\right\} \qquad \text{(A-19)}$$

Although tedious, evaluation of this involves terms whose magnitudes are easy to handle. S does not get to be very large---as an example, N = 100 and $\Lambda = 187.248$ ($\Rightarrow P = 10^{-12}$) yield S = 2.096. Then, P can be rather readily evaluated from (A-18). The only approximation that need be made is Stirling's for the factorial. Using the form

$$\ln (N!) \approx (N+1/2) \ln N - N + 1/2 \ln (2\pi) + \frac{1}{12N} - \frac{1}{360N^3}$$

results in an accuracy of at least 9 places for $N > 7$.

To invert, we wish to calculate the root of

$$f(\Lambda) \triangleq P(N,\Lambda) - P = 0$$

Newton's method for the solution of this involves iterations of the form

$$\Lambda_{i+1} = \Lambda_i - \frac{f(\Lambda_i)}{f'(\Lambda_i)}$$

Since (by original definition)

$$P(N,\Lambda) = \int_{\Lambda}^{\infty} \frac{e^{-t}\, t^{N-1}}{(N-1)!}\, dt$$

we have

$$\frac{d}{d\Lambda} P(N,\Lambda) = -\frac{e^{-\Lambda}\, \Lambda^{N-1}}{(N-1)!}$$

The iteration then reduces to

$$\Lambda_{i+1} = \Lambda_i + S(N,\Lambda_i) - A(N,\Lambda_i) \qquad \text{(A-20)}$$

where

$$A(N,\Lambda) = \frac{P\ (N-1)!}{e^{-\Lambda}\ \Lambda^{N-1}} \qquad \text{(A-21)}$$

A (N,Λ) is readily calculated and is of the same order of magnitude as S. Thus, evaluation of equation (A-20) is a practical technique for inversion. Using a good starting value, such as given by the solution of (A-11) or (A-13), only a few iterations are required to reach a solution.

APPENDIX E

DETECTION OF A MESSAGE REDUNDANTLY TRANSMITTED OVER A FADING CHANNEL

PART 2 - INTERCEPT ANALYSES

TECHNICAL NOTE 76-03

DETECTION OF A MESSAGE REDUNDANTLY TRANSMITTED OVER A FADING CHANNEL

PART 2 - INTERCEPT ANALYSES

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DETECTION OF A MESSAGE REDUNDANTLY TRANSMITTED OVER A FADING CHANNEL Part 2 - INTERCEPT ANALYSES

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ABSTRACT

This report is a logical companion to Technical Note 76-01, "Detection of a Message Redundantly Transmitted Over a Fading Channel". In this report, we analyze the performance of a would-be interceptor, comparing his ability to detect the message versus the capability of the intended receiver, using a measure termed "recognition differential". Performance curves for two interceptors, which form upper and lower bounds to the performance of any interceptor, are presented.

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SECTION 1

INTRODUCTION AND SUMMARY

In a companion report (Ref. 1) we presented a communications problem and an approach to its solution. Briefly, the requirement was to transmit with high reliability the logical equivalent of one bit of information. The constraints included a lack of synchronization between transmitter and receiver, the use of a fading dispersive channel and the need to be covert. A message structure and optimum receiver processing were proposed. Idealized performance results (expressed in terms of detection probability versus signal-to-noise ratio) were derived. Modifications to these results to account for the effects of the dispersive nature of the channel were also discussed.

In this paper, we analyze the rest of the problem---i.e., the covertness of the proposed signalling scheme. Two potential interceptors are described and their performances are derived in a manner analogous to that for the intended receiver. These levels of performance are then contrasted against that of the intended receiver. The difference is termed recognition differential and is a measure of the covertness of the chosen message structure.

As explained in our earlier report, the channel imposed distortions are such that incoherent processing is all that is available to the intended receiver. This forces the use of techniques similar to those which would be employed by an unauthorized receiver, and much of the processing gain available in more conventional LPI systems is lost. Here, we show that relatively long (e.g., 100 tones) messages are needed to achieve a reasonable (6 to 10 dB) value of recognition differential.

SECTION 2

STATEMENT OF THE PROBLEM

2.1 THE MESSAGE

In this section we summarize the format of the transmitted message as detailed in Ref. 1. The message is assumed to consist of a total of N tones, where N is a design parameter. Each tone has the same basic pulse shape and is centered in a unit area of duration T and width of Δf. Time is correspondingly quantized into intervals of length T, frequency into intervals of width Δf, forming a time-frequency matrix. The message tones are located in the matrix according to a frequency-hopping pattern.

Specifically, we consider a message of length N located within an NxN matrix such that at any time only one frequency is used, and each frequency is used at only one time. The particular choice of tone locations used in the matrix is controlled by the message pattern or key. This is assumed known to the intended receiver but not to an unauthorized or intercept receiver. It is this knowledge which enables the intended receiver to detect a message at a lower received signal-to-noise level than the interceptor can, and it is this "processing gain" which (as a function of the message length N) we wish to examine.

2.2 ASSUMPTIONS

Several assumptions regarding the processing done by a would-be interceptor are briefly stated in this section. In general, these parallel those made concerning the authorized receiver in our previous analysis. Following this, we describe the general form that our analyses will take.

Due to the Rayleigh fading channel, the received signal tones are samples of a complex Gaussian process. It is assumed that the time and/or frequency separation between active tones in the message pattern is sufficient to ensure that they are all mutually independent. It is further assumed that they are all received at the same energy level. The background noise is taken to be additive Gaussian, independent and identically distributed across the time-frequency matrix.

The noise variance is σ^2, that of the signal is σ_s^2; their ratio is the signal-to-noise ratio denoted γ, which is assumed constant for all of the received message tones. Letting u be the response out of an arbitrary time-frequency resolution cell, we then have two basic sub-hypotheses:

$$\left.\begin{array}{l} h_o \ : u \in \text{complex } N(0,\sigma^2) \ , \text{ noise only} \\ h_1 \ : u \in \text{complex } N(0,\sigma^2/\alpha) \ , \text{ signal + noise} \\ \text{with } \ \alpha \triangleq \dfrac{1}{1+\gamma} \end{array}\right\} \qquad (2\text{-}1)$$

Although the above may seem an obvious starting point, it tacitly makes several "worst-case" assumptions regards the interceptor. First, it assumes that he is forming the proper time-frequency matrix at his receiver. That is he knows the pulse length and width and processes his input signal accordingly. Further, he is synchronized

in both time and frequency with the message and there is no run-out due to Doppler shift. We will also assume that the interceptor knows the total number of active tones (N) and the dimensions of the matrix containing them.

The assumption of synchronization is one that was also made in our earlier analysis. There, we described how to model the more realistic unsynchronized situation by decreasing the effective SNR per tone. That discussion applies equally well to the intercept receiver, in fact, when comparing SNRs for the two receivers, this extra factor washes out.

Whether or not an interceptor would in reality know the form of the message (T; Δf; N) is not within the scope of this study. We choose these assumptions using the standard rationalization behind any worst-case analysis. However, in interpreting the results, their basic pessimistic nature should not be forgotten.

For simplicity of description, we will picture the interceptor as forming and storing* an NxN array of responses exactly as the authorized receiver would. Incoherent (square-law) processing will always be used, so only the magnitude-squared of each resolution cell need be saved. Only two cases will be of interest. The noise-only hypothesis (H_o) assumes that none of the N tones in the message are within the matrix currently being examined; the signal hypothesis (H_1) assumes that they all are.

*As explained in Section 2.4, this is not really necessary.

According to some algorithm, the interceptor will process the responses in his received time-frequency matrix and form therefrom a decision variable. Just as for the authorized receiver, this will be compared against a threshold to determine whether a message is present or not. The setting of this threshold will be chosen to yield the desired per-decision false-alarm probability (P_{FA}). The signal-to-noise ratio required by the interceptor to achieve probability of detection = 0.5 is computed. This value is then compared to that required by the authorized receiver to similarly have a fifty percent detection probability. The difference (in dB) between these signal-to-noise ratios is termed "recognition differential" and is a direct measure of the processing gain that the intended receiver has over an interceptor.

In general, an interceptor's received SNR will differ from that of the authorized receiver. Three possible causes are

1) Different range from the transmitter.

2) Different bearing relative to the transmitted beam center (MRA-Main Response Axis).

3) Different directionality index of the receiving array.

Any of these can result in a received SNR greater or less than that of the intended receiver. Here, we do not concern ourselves with particular combinations of these three possibilities---we merely derive the required net effect. Given a particular operational scenario, recognition differential may then be expressed in terms of one of these parameters. As an example, if an interceptor were presumed to use a receiving array of the same directivity and was at the same angle relative to MRA as the intended receiver, then the recognition differential could be equated to a "range advantage" for the intended receiver.

2.3 POSSIBLE INTERCEPTORS

For the intended receiver, it was obvious that only those cells in the time-frequency matrix (as determined by the key) corresponding to signals should be considered in the formation of a decision as to the presence of a message. The interceptor does not have such an obvious strategy, in fact, several forms of processing could equally well be postulated, each representing differing degrees of sophistication and/or knowledge of the message structure. We briefly discuss some of these and then select two for analysis.

The most simplistic form of intercept analysis examines only one time slice or frequency cell. As discussed in Section 2 of our earlier report, this leads to a message structure of at most one tone per time or frequency in order to minimize the performance for this interceptor. Although an interceptor may also use more complex strategies, we take this as a baseline system against which covertness must be considered. Hence, we choose our basic message as indicated to optimize performance against this basic interceptor. In order to evaluate more significant threats, we now consider ways in which an interceptor can process over the entire NxN matrix in an attempt to detect the presence of a message.

We will choose two interceptors which may be construed as bounding the performance of any sophisticated yet realistic threat. The first is a traditional energy detector (denoted ED). This interceptor simply sums all of the N^2 responses in the matrix. This decision variable is then compared to a threshold. Although the ED is a sophisticated processor in that it makes a decision after examining the entire received NxN

matrix, it represents the most elementary form of processing such a matrix. Thus, we view its performance as being a lower bound to the performance of an arbitrary interceptor.

We now additionally assume that the interceptor knows that the message is structured so that only one tone is transmitted at any time. With this knowledge, a better form of processing is possible. Specifically, for each time slice, the greatest received power level in the N frequency cells is selected and saved. The sum of these, accumulated over N time slices, forms the decision variable. We refer to this as the "greatest of" receiver, denoted GOF.

Knowing that only one tone is transmitted at a time defines N^N possible patterns. Further specifying (which we do not consider here) that no frequencies are repeated lowers this number to N!. An optimum interceptor would form a decision variable corresponding to each of these possible message patterns. Without possession of the key, the interceptor must consider each of these as representing a possible message. The most likely message is simply the one with the greatest associated decision variable. An optimum interceptor would test this most likely message to decide if any message were present. But, by maximizing the likelihood ratio in each time slice, the GOF directly extracts this maximum likelihood term. Thus, the GOF is equivalent to the optimum form of processing, and is clearly within the constraints of practicality. We consider the GOF as representing the optimum interceptor, and we interpret its performance as being an upper bound to that of any interceptor.

Having concocted an interceptor optimized against our message structure, it is tempting to consider changing the message structure. For instance, if in one time slice two tones were transmitted and then none in the next, the GOF would clearly suffer a decrease in performance. However, such reasoning is circular as, having postulated this new message, one could in turn define a new intercept strategy, optimized against it. Thus, we fix our message structure as described and merely analyze how well these two interceptors perform against it.

We might further consider other intercept strategies. One such might be termed "threshold counting." Here, each of the N^2 responses is compared against a fixed threshold set by the noise variance and the number of exceedances is counted. This total is then compared against another threshold and a decision as to the presence of a message is made. This technique is related to the cognitive processes of a trained operator who is viewing the responses on a screen and looks for an unusually large number of "blips." It is probably less useful as a model of a technique which would be implemented purely electronically. However, our subsequent analysis shows that our upper and lower bounds to the performance of an arbitrary interceptor are relatively close. Thus, we do not analyze in detail the threshold counter or any other possible intercept scheme.

2.4 PROCESSING CONSIDERATIONS

In this section we discuss briefly some significant differences between an interceptor and an authorized receiver. The authorized receiver has an advantage in that he alone possesses the message key. However, this also penalizes him. As explained in Ref. 1, he must examine the received time-frequency matrix every time slice in order to detect the message. Further, multiple Doppler hypotheses are required to properly align the received tones with the pattern.

Since the interceptor does not have the pattern, he does not care about the fine alignment of his time-frequency matrix with the message. Not only need he not form multiple Doppler hypotheses, he need not examine every matrix. For instance, examination of only every tenth matrix results in missing at most 5 tones. For a message consisting of 128 tones, this represents a loss in signal energy of only .2 dB, which can be considered as being negligible. Thus, the interceptor need make decisions much less frequently than the intended receiver. We also note that, for both the ED and GOF, successive decision variables will be highly correlated.

The significance of all this is that, in order to have a false-alarm interval the same as that for the intended receiver (which seems the most reasonable manner in which to compare their performance), the interceptor can operate at a higher false-alarm probability. This is equivalent to a lowering of his threshold, thereby increasing his detection probability. The difference between operating P_{FA}s can be one or two orders of magnitude. In our comparisons, we will assume that the intended receiver is operating at $P_{FA} = 10^{-6}$ and the interceptor at 10^{-5}.

SECTION 3

OUTLINE OF ANALYSES

Our analysis of an intercept receiver is somewhat different in approach from our earlier analysis of the performance of the intended receiver. It involves initially making two assumptions (valid for large N) and then determining correction terms for smaller N. As this approach is common to our discussion of both the Energy Detector and the Greatest of interceptors, we outline it here.

We consider two idealized hypotheses, H_0 and H_1, corresponding respectively to receiving noise only and noise plus all N message tones. The calculated decision variable is, in general, denoted as v. To emphasize the dependence of the statistics of v upon the particular hypothesis, we will write v_j with j equal to either 0 or 1. Similarly, where it is necessary to explicitly state the number of terms used in the combination forming v (i.e., the order of the random variable) this shall be represented as an argument, e.g., v_0 (N^2).

We note that for the ED and GOF, v_0 is formed as the sum of many independent and identically distributed (iid) random variables. We thus approximate v_0 by a normal random variable with appropriate mean and variance. This allows us to calculate a decision threshold which closely approximates the ideal threshold corresponding to a specified P_{FA} as follows:

First, the mean (μ) and variance ($\sigma_0{}^2$) of v_0 are calculated. From a tabulation of the standard (i.e., zero mean and unit variance) normal probability function, such as in Table 3.1, the value λ

TABLE 3-1

NORMAL PROBABILITY FUNCTION FOR EXTREME VALUES OF Q (λ)

$$P_{FA} = Q(\lambda) \triangleq \int_{\lambda}^{\infty} z(t)dt \quad ; \quad z(t) \in N(0,1)$$

$$P_{FA} = 10^{-p}$$

p	λ
2	2.32635
3	3.09023
4	3.71902
5	4.26489
6	4.75342
7	5.19934
8	5.61200
9	5.99781

corresponding to P_{FA} is found. Then, the approximate threshold for the interceptor is obtained as

$$\Lambda = \mu + \lambda\sigma_0 \tag{3-1}$$

With the threshold established as just described we next consider the signal present hypothesis H_1. We determine the required SNR to achieve an intercept probability of 0.5. This point corresponds to the SNR for which the median of v_1 equals the threshold. We make our second approximation, which is that the mean of v_1 and the median of v_1 coincide. This relationship is strictly valid for symmetrical distributions and is closely approximated for the high-ordered distributions of interest here. To have a fifty percent detection probability is thus equivalent to specifying that

$$E[v_1] \approx \Lambda \tag{3-2}$$

The expected value of v_1 may be expressed as a function of the signal-to-noise ratio. This allows us to invert Equation 3-2 to solve for SNR as a function of Λ. Using Equation 3-1, we then eliminate Λ, resulting in

$$E[v_1] = E[v_0] + \lambda\sqrt{\mathrm{var}[v_0]} \tag{3-3}$$

This is solved for SNR as a function of N, completing our initial analysis.

We wish to produce, as our measure of performance, a curve showing required SNR vs message length N. The results determined from Equation 3-3 are only asymptotically correct for large N. By more precisely calculating points for smaller N, we can modify this curve to eliminate the errors introduced by our assumptions. By their definition, both the ED and the GOF are equivalent to the intended receiver for the

degenerate case of $N = 1$. Thus, the exact performance is already known for $N = 1$. More accurate calculations for a few other values of N then allows a smooth curve to be fitted to the asymptotic result.

Of our two assumptions, the first represents the greatest source of error. Here, the central limit theorem is used to approximate the distribution of v_0 in its tails, and it is here that the limiting form is a poor approximation for small N. More detailed discussions of accuracy and corrections are presented in the particular analysis following.

The second assumption is not regarded as introducing as significant an error. For one reason, it is a weaker assumption than the first. The condition that the mean and median coincide is satisfied by any random variable with density symmetric about its mean,* whether it be normal or not. Further, the exact shape of the tails of the density of v_1 have little effect on the desired condition. As an example, for a chi-squared density of order greater than 21, the difference between the mean and the median is less than .1 standard deviation.

*Also, some non-symmetric examples can be concocted.

SECTION 4

THE ENERGY DETECTOR

The energy detector forms a decision variable v_j by adding the magnitude-squared of all the responses in the matrix. This is normalized by the factor $2\sigma^2$, where σ^2 is the (presumed known) variance of the noise process and then compared against a fixed threshold Λ. The decision variable is in general denoted by v, more specifically, by

$$v_j(K) \triangleq \frac{1}{2\sigma^2} \sum_{i=1}^{K} \left|u_i\right|^2 \Bigg|_{H_j} \tag{4-1}$$

where the subscript j specifies the hypothesized distribution of the responses $\{u_i\}$ and the argument K is the number of terms used in forming v.

For an NxN matrix, the two hypotheses are given by

$$\left.\begin{array}{l} H_0\text{: all } N^2 \text{ terms are purely noise} \\ H_1\text{: } (N^2-N) \text{ terms are noise; N are signal plus noise} \end{array}\right\} \tag{4-2}$$

Under H_0, the $\{u_i\}$ are i.i.d. and $v_0(N^2)$ has a standard Gamma distribution of order N^2. We thus have

$$\left.\begin{array}{l} E\left[v_0(N^2)\right] = N^2 \\ \mathrm{Var}\left[v_0(N^2)\right] = N^2 \end{array}\right\} \tag{4-3}$$

Under H_1, the summation indicated in Equation (4-1) may be re-written as

$$v_1(N^2) = \frac{1}{2\sigma^2} \sum_{i=1}^{N^2-N} \left|u_i\right|^2 \Bigg|_{h_0} + \frac{1}{2\sigma^2} \sum_{i=1}^{N} \left|u_i\right|^2 \Bigg|_{h_1}$$

where the sub-hypotheses h_0 and h_1 determining the distribution of the

individual u_i are as defined in Equation (2-1). We then have two independent summations. The first is simply a standard Gamma variate of order (N^2-N). The second is also a Gamma variate, or order N, but is non-standard---i.e., it has a shape parameter not equal to unity. Although $v_1(N^2)$ is the sum of two Gamma variates, since they have differing shape parameters, the reproductive property of the Gamma distribution does not hold. Thus, their sum is not a Gamma variate and its density must be determined by convolution.

By normalizing the second sum by the factor $1/\alpha$, it can be written in terms of a standard Gamma variate. Then, $v_1(N^2)$ may be expressed as a function of two variables formed under H_0,

$$v_1(N^2) = v_0(N^2-N) + \frac{1}{\alpha} v_0(N) \tag{4-4}$$

From this, we directly have the moments of v_1:

$$\begin{aligned} E\left[v_1(N^2)\right] &= E\left[v_0(N^2-N)\right] + \frac{1}{\alpha} E\left[v_0(N)\right] \\ &= (N^2-N) + (1+\gamma)\ (N) \\ &= N^2 + \gamma N \end{aligned} \tag{4-5}$$

and,

$$\begin{aligned} \text{var}\left[v_1\ (N^2)\right] &= (N^2-N) + (1+\gamma)^2\ (N) \\ &= N^2 + 2\gamma N + \gamma^2 N \end{aligned} \tag{4-6}$$

or,

$$E\left[v_1\ (N^2)\right] = E\left[v_0(N^2)\right] + \gamma N \tag{4-7}$$

$$\text{var}\left[v_1\ (N^2)\right] = \text{var}\ \left[v_0(N^2)\right] + 2\gamma N + \gamma^2 N \tag{4-8}$$

Inserting (4-7) and (4-3) into Equation (3-3), we find the SNR required for the ED to achieve P_D=.5 to simply be given by

$$\gamma = \lambda \tag{4-9}$$

This rather surprising result states that, as long as N is large enough for our approximations to be valid, γ is independent of N.

As explained in Section 2.4, we shall specifically consider an interceptor operating at a false-alarm level of 10^{-5}. We then have, from Table 3-1,

$$\gamma = 4.265 \quad \text{or} \quad \gamma = 6.30 \text{ dB} \tag{4-10}$$

This is shown as the solid line in Figure 4-1.

This result is asymptotically correct for large N. We now determine more exact results for small N. First, we note that the primary source of error is in the setting of the threshold using the values in Equation (4-3). For the ED, the exact threshold $\Lambda(N^2)$ is easily enough determined. Using the tables given in Reference 2, values of the threshold Λ yielding a P_{FA} of 10^{-5} are found. The requisite SNR is then calculated using Equations (3-2) and (4-5), viz.,

$$N^2 + \gamma N = \Lambda(N^2) \tag{4-11}$$

These results are shown as the dashed line in Figure 4-1.

Finally, we examine the error introduced by equating the median of $v_1(N^2)$ to its mean. This necessitates determining the density of v_1 by convolving the densities of the random variables indicated in Equation (4-4). This is then integrated to determine the fiftieth percentile point. Equating this to the exact threshold will eliminate all approximations and the determined answer will be an exact level of performance for the ED.

For N=1, the exact SNR for P_D=.5 is the same as for the intended receiver and is readily obtained from our earlier analysis.

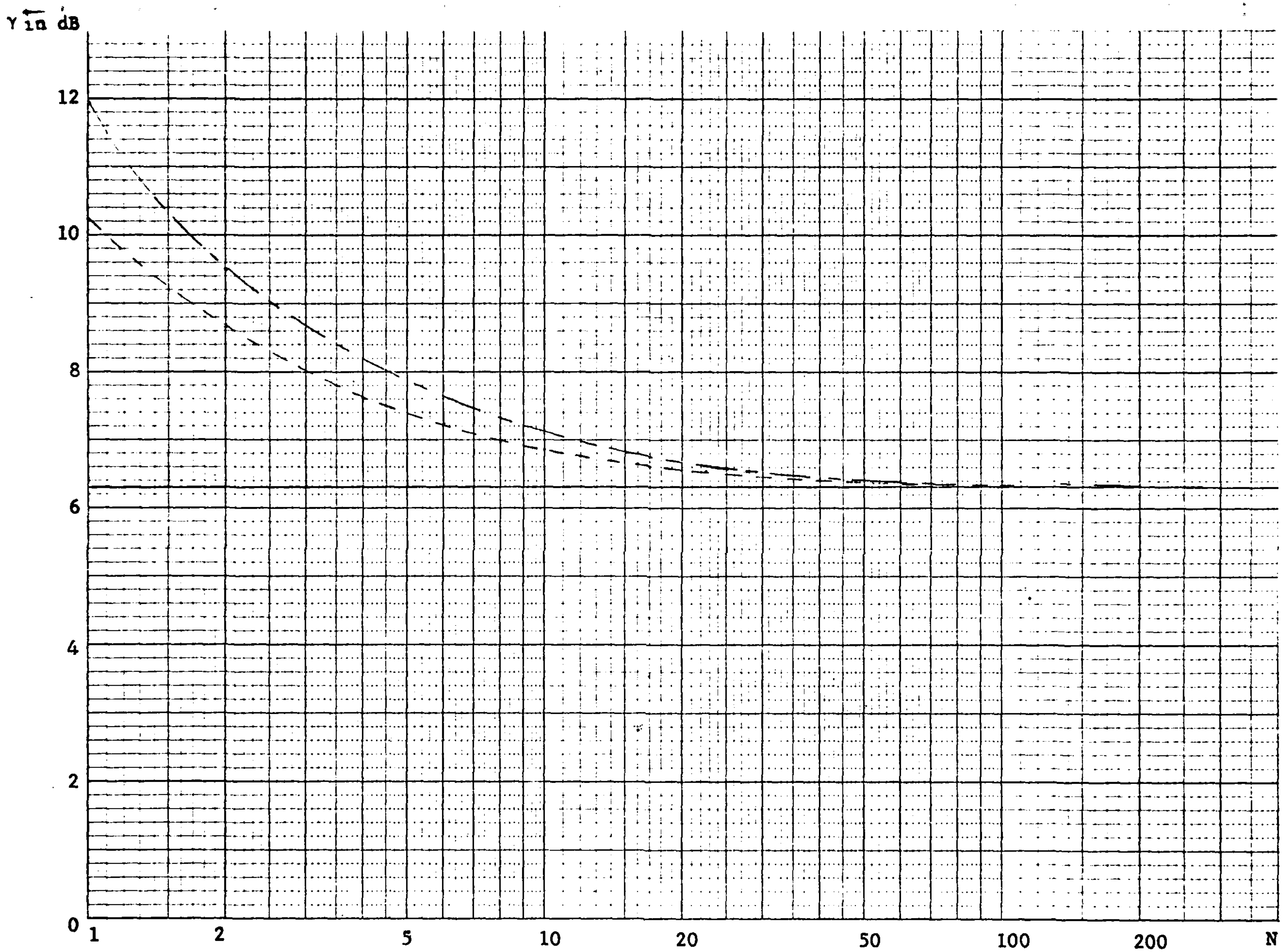

FIGURE 4-1 PERFORMANCE OF THE ED: SNR FOR $P_D = .5$ @ $P_{FA} = 10^{-5}$

From Figure 4-3 of Reference 1, we have γ=11.9 dB. For N=2, the convolution and integration are feasible, although already awkward. The required SNR is calculated to be 9.50 dB. Exact values could be found for larger N, but the effort does not appear justified, as the corrections are small. A sufficiently smooth curve can be fitted through these two exact results to the asymptotic result. This is the dashed and dotted curve in Figure 4-1.

SECTION 5

THE GREATEST OF RECEIVER

Out of each time slice, the "greatest-of" receiver extracts the greatest (magnitude-squared) response. This is done for the appropriate number of time slices. The selected terms are summed and normalized by the factor $2\sigma^2$ to form the decision statistic. This is then compared against a fixed threshold Λ.

As we did before, we will denote the decision statistic as v, or, more specifically as $v_j(K)$, where j specifies the hypothesis and K, the number of terms summed to form v. Whereas for the ED v was formed by directly adding the $\left\{ |u_i|^2 \right\}$, we now form v as the sum of random variables which are functions of the $\left\{ |u_i|^2 \right\}$. These functions represent the "greatest of" operation. We denote the individual terms by w, or more specifically, by

$$w_j(K') \triangleq \frac{1}{2\sigma^2} \max \left\{ |u_i|^2, \ 1 \leq i \leq K' \right\} \Bigg|_{H_j} \tag{5-1}$$

where the subscript j specifies the hypothesis controlling the distribution of the $\{u_i\}$ and the argument K' is the number of responses from which the greatest is extracted. Note that for convenience in the analysis the normalizing factor $2\sigma^2$ (where σ^2 is the known noise variance) is applied directly to the $\{w\}$. The decision statistic is then defined as

$$v_j \ (K) \triangleq \sum_{i=1}^{K} \left(w_j(K') \right)_i \tag{5-2}$$

The arguments K and K' correspond respectively to the number of time slices and the number of frequency cells in the time/frequency matrix. As described in Section 2.2, we are considering the special

case of an NxN matrix, hence, K and K' are equal. In a more general formulation, however, one would need to properly distinguish them. Here we set them both to N.

As we are assuming the signal and noise terms throughout the matrix to all be mutually independent, we clearly have the $\{u_i\}$ defining w in Equation (5-1) all mutually independent; similarly so are the $\{w\}$ defining v in Equation (5-2). The distribution of the individual $\{u_i\}$ in (5-1) is determined by the choice of hypothesis:

$$\left.\begin{array}{l} H_0\text{: all N of the } \{u_i\} \text{ are noise } (h_0) \\ H_1\text{: (N-1) of the } \{u_i\} \text{ are noise } (h_0)\text{; one is signal + noise} \end{array}\right\} \quad (5\text{-}3)$$

where the sub-hypotheses h_0 and h_1 are defined in Equation (2-1). Thus, within one time slice, the $\{u_i\}$, although independent, are not necessarily identically distributed. However, from one time slice to the next, the individual greatest of terms appearing in the summation of (5-2) are iid.

We first determine the probability density function of w under H_0 and H_1. This is done by finding the distribution function and differentiating.

Examining an individual response under the two sub-hypotheses of Equation (2-1), we calculate the probability that this response is less than some value w. The normalized magnitude squared is simply an exponentially distributed random variable, hence

$$P\left\{\frac{1}{2\sigma^2}\left|u_i\right|^2 < w\right\} = \begin{cases} 1-e^{-w}, & h_0 \\ 1-e^{-\alpha w}, & h_1 \end{cases} \quad (5\text{-}4)$$

By the assumed independence of the $\{u_i\}$, we have

$$P\left\{\max\left\{\frac{1}{2\sigma^2}\left|u_i\right|^2\right\} < w\right\}\Bigg|_{H_j} = \prod_{i=1}^{N} P\left\{\frac{1}{2\sigma^2}\left|u_i\right|^2 < w\right\}\Bigg|_{H_j} \tag{5-5}$$

This specifies the distribution function of $w_j(N)$, which we denote as $F_{j,N}(w)$. Substitution into Equation (5-5) from Equations (5-4) and (5-3) then yields:

$$F_{0,N}(w) = (1 - e^{-w})^N \tag{5-6}$$

$$F_{1,N}(w) = (1 - e^{-w})^{N-1}\,(1 - e^{-\alpha w}) \tag{5-7}$$

The density functions, denoted by f rather than F are readily found to be:

$$f_{0,N}(w) = N(1 - e^{-w})^{N-1}(e^{-w}) \tag{5-8}$$

$$f_{1,N}(w) = (N - 1)(1 - e^{-w})^{N-2}e^{-w}(1 - e^{-\alpha w}) + \alpha e^{-\alpha w}(1 - e^{-w})^{N-1} \tag{5-9}$$

The density of w under H_1 may alternatively be expressed in terms of the density under H_0:

$$\begin{aligned} f_{1,N}(w) &= f_{0,N-1}(w) - (N-1)(1 - e^{-w})^{N-2}e^{-\alpha w}e^{-w} \\ &\quad + \alpha(1 - e^{-w})^{N-1}e^{-\alpha w} \end{aligned} \tag{5-10}$$

Next, we evaluate the moments of w. Details are presented in Appendix A. We first show that

$$E[w_0(N)] = \sum_{k=1}^{N} \binom{N}{k} (-1)^{k-1} \left(\frac{1}{k}\right)$$

$$E[w_0^2(N)] = 2\sum_{k=1}^{N} \binom{N}{k} (-1)^{k-1} \left(\frac{1}{k}\right)^2$$

$$E[w_1(N)] = E[w_0(N-1)] + \sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k \left(\frac{1}{k+\alpha}\right)$$

$$E[w_1^2(N)] = E[w_0^2(N-1)] + 2\sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k \left(\frac{1}{k+\alpha}\right)^2$$

Simplification of these leads to the following results. The mean and variance of w_0 are unexpectedly simple and elegant in form.

$$E\left[w_0(N)\right] = \sum_{k=1}^{N} \left(\frac{1}{k}\right) \tag{5-11}$$

$$\text{var}\left[w_0(N)\right] = \sum_{k=1}^{N} \left(\frac{1}{k}\right)^2 \tag{5-12}$$

$$E\left[w_1(N)\right] = E\left[w_0(N-1)\right] + B(\alpha,N) \tag{5-13}$$

It does not appear possible to simplify the expression for the variance of w_1, however, we do not really need to know this.

The expression for the mean of w_0 is the well known harmonic series, which is easily shown to be divergent. Thus,

$$\lim_{N\to\infty} E\left[w_0(N)\right] = \infty$$

The variance is a lesser known series; it converges and can be succinctly expressed in the limit as

$$\lim_{N\to\infty} \text{var}\left[w_0(N)\right] = \frac{\pi^2}{6}$$

The function $B(\alpha,N)$ appearing in the mean of w_1 is the Beta function, defined by,

$$B(\alpha,N) = B(N,\alpha) = \int_0^1 (1-t)^{N-1}\, t^{\alpha-1}\, dt$$

This may be expressed in terms of Gamma functions,

$$B(\alpha,N) = \frac{\Gamma(\alpha)\,\Gamma(N)}{\Gamma(\alpha+N)} \tag{5-14}$$

which, for computational purposes, is written as

$$B(\alpha,N) = \frac{(N-1)}{(N-1+\alpha)} \times \frac{(N-2)}{(N-2+\alpha)} \times \cdots \times \frac{1}{(1+\alpha)} \times \frac{1}{\alpha} \tag{5-15}$$

Values of $B(\alpha,N)$, presented against γ in dB (from whence α is determined), are shown in Table 5-1.

TABLE 5-1

VALUES OF B(N,α)

GAMMA	ALPHA	BETA (N,ALPHA) N = 1	2	4	8	16	32	64	128
-10.00	0.90909	1.100	0.576	0.304	0.161	0.086	0.046	0.024	0.013
-9.00	0.88818	1.126	0.596	0.319	0.171	0.092	0.050	0.027	0.014
-8.00	0.86319	1.158	0.622	0.337	0.184	0.101	0.055	0.030	0.017
-7.00	0.83366	1.200	0.654	0.361	0.201	0.112	0.063	0.035	0.020
-6.00	0.79924	1.251	0.695	0.392	0.223	0.128	0.073	0.042	0.024
-5.00	0.75975	1.316	0.748	0.433	0.253	0.148	0.087	0.052	0.030
-4.00	0.71525	1.398	0.815	0.465	0.292	0.177	0.107	0.065	0.040
-3.00	0.66614	1.501	0.901	0.553	0.344	0.215	0.135	0.085	0.054
-2.00	0.61314	1.631	1.011	0.643	0.414	0.269	0.175	0.114	0.075
-1.00	0.55731	1.794	1.152	0.760	0.509	0.343	0.232	0.157	0.107
0.00	0.50000	2.000	1.333	0.914	0.637	0.447	0.315	0.222	0.157
1.00	0.44269	2.259	1.566	1.117	0.809	0.591	0.433	0.318	0.234
2.00	0.38686	2.585	1.864	1.383	1.042	0.791	0.603	0.460	0.352
3.00	0.33386	2.995	2.246	1.732	1.355	1.067	0.844	0.668	0.530
4.00	0.28475	3.512	2.734	2.185	1.771	1.444	1.182	0.969	0.794
5.00	0.24025	4.162	3.356	2.774	2.321	1.954	1.649	1.394	1.179
6.00	0.20076	4.981	4.148	3.533	3.043	2.634	2.286	1.986	1.727
7.00	0.16634	6.012	5.154	4.509	3.982	3.532	3.141	2.796	2.490
8.00	0.13681	7.310	6.430	5.756	5.195	4.707	4.273	3.883	3.530
9.00	0.11162	8.943	8.044	7.344	6.753	6.229	5.756	5.322	4.923
10.00	0.09091	11.000	10.083	9.361	8.742	8.186	7.677	7.203	6.761
11.00	0.07359	13.589	12.658	11.916	11.273	10.689	10.147	9.637	9.155
12.00	0.05935	16.849	15.905	15.147	14.483	13.874	13.304	12.762	12.245
13.00	0.04773	20.953	19.998	19.226	18.545	17.915	17.320	16.750	16.202
14.00	0.03829	26.119	25.156	24.372	23.676	23.029	22.413	21.819	21.245
15.00	0.03065	32.623	31.653	30.859	30.151	29.489	28.856	28.242	27.645
16.00	0.02450	40.811	39.835	39.034	38.316	37.641	36.993	36.364	35.748
17.00	0.01956	51.119	50.138	49.331	48.604	47.920	47.260	46.617	45.986
18.00	0.01560	64.096	63.111	62.299	61.566	60.874	60.204	59.549	58.905
19.00	0.01243	80.433	79.445	78.628	77.890	77.191	76.514	75.850	75.196
20.00	0.00990	101.000	100.010	99.190	98.447	97.743	97.060	96.388	95.725
21.00	0.00788	126.893	125.900	125.078	124.332	123.623	122.935	122.257	121.588
22.00	0.00627	159.489	158.496	157.671	156.922	156.210	155.517	154.835	154.160
23.00	0.00499	200.526	199.531	198.705	197.954	197.239	196.543	195.857	195.177
24.00	0.00397	252.189	251.193	250.365	249.612	248.895	248.197	247.508	246.824
25.00	0.00315	317.228	316.231	315.402	314.648	313.929	313.229	312.537	311.852
26.00	0.00251	399.107	398.110	397.280	396.525	395.805	395.102	394.409	393.721
27.00	0.00199	502.187	501.189	500.359	499.603	498.882	498.178	497.483	496.793
28.00	0.00158	631.957	630.959	630.128	629.371	628.649	627.944	627.247	626.557
29.00	0.00126	795.328	794.330	793.498	792.741	792.018	791.313	790.615	789.922
30.00	0.00100	1001.000	1000.001	999.169	998.411	997.688	996.982	996.283	995.589

From the moments of w, we determine the moments of v. Since v is the sum of N of the w, all of which are i.i.d., this simply involves multiplying by N. Values of the mean, variance and standard deviation of w_0 (labelled "single") and of v_0 (labelled "N Times") are shown in Table 5-2.

Our initial approximation to the SNR required by the GOF to achieve $P_D = .5$ is given by Equation (3-3). Substituting into this, we have

$$N\,E\left[w_0(N)\right] + \lambda\sqrt{N\cdot\mathrm{var}\left[w_0(N)\right]} = N\,E\left[w_1(N)\right] = N\left\{E\left[w_0(N-1)\right] + B(\alpha,N)\right\}$$

where the last equality results from Equation (5-13). Rearranging,

$$N\,B(\alpha,N) = N\left\{E\left[w_0(N)\right] - E\left[w_0(N-1)\right]\right\} + \lambda\sqrt{N\cdot\mathrm{var}\left[w_0(N)\right]}$$

Using Equation (5-11), this reduces to

$$B(\alpha,N) = \frac{1}{N} + \frac{\lambda\sqrt{\mathrm{var}\left[w_0(N)\right]}}{\sqrt{N}} \tag{5-16}$$

This is the equation we solve to determine γ.

As before, we set $\lambda = 4.26489$ for a false-alarm probability of 10^{-5}. Using Tables 5-1 and 5-2 (or more finely granulated versions thereof) we then readily determine γ as a function of N. This is plotted as the solid line in Figure 5-1.

Asymptotic performance is not as readily determined as for the energy detector. However, we do note that for false-alarm probabilities less than .084,

$$\lim_{N\to\infty} \gamma(N) > 1 \tag{5-17}$$

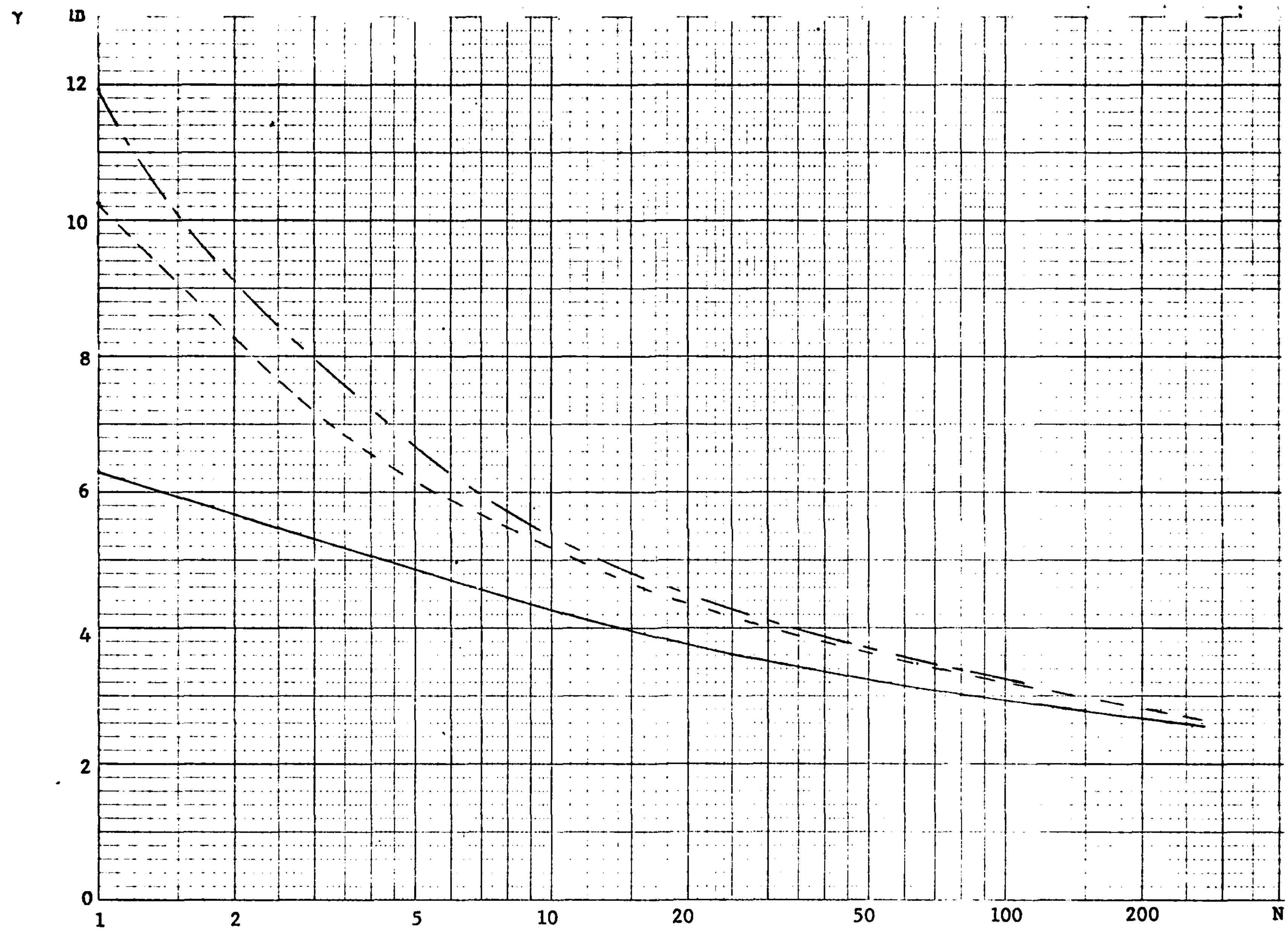

FIGURE 5-1 PERFORMANCE OF THE GOF: SNR FOR $P_D = .5$ @ $P_{FA} = 10^{-5}$

TABLE 5-2

MOMENTS OF $W_0(N)$ AND $V_0(N)$

N	MEAN	SINGLE VARIANCE	STD DEV	MEAN	N TIMES VARIANCE'	STD DEV
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.50000	1.25000	1.11803	3.00000	2.50000	1.58114
3	1.83333	1.36111	1.16667	5.50000	4.08333	2.02073
4	2.08333	1.42361	1.19315	8.33333	5.69444	2.38630
5	2.28333	1.46361	1.20980	11.41667	7.31806	2.70519
6	2.45000	1.49139	1.22122	14.70000	8.94833	2.99138
7	2.59286	1.51180	1.22955	18.15000	10.58258	3.25309
8	2.71786	1.52742	1.23589	21.74286	12.21938	3.49562
9	2.82897	1.53977	1.24067	25.46072	13.85791	3.72262
10	2.92897	1.54977	1.24490	29.28968	15.49768	3.93671
20	3.59774	1.59616	1.26339	71.95480	31.92327	5.65007
30	3.99499	1.61215	1.26970	119.84962	48.36451	6.95446
40	4.27854	1.62024	1.27289	171.14174	64.60977	8.05045
50	4.49921	1.62513	1.27481	224.96028	81.25665	9.01425
60	4.67987	1.62841	1.27609	280.79227	97.70434	9.88455
70	4.83284	1.63075	1.27701	338.29865	114.15250	10.68422
80	4.96548	1.63251	1.27770	397.23840	130.60095	11.42808
90	5.08257	1.63388	1.27823	457.43140	147.04962	12.12640
100	5.18738	1.63498	1.27866	518.73779	163.49840	12.78665
110	5.28224	1.63588	1.27902	581.04584	179.94730	13.41444
120	5.36887	1.63664	1.27931	644.26422	196.39626	14.01414
130	5.44859	1.63727	1.27956	708.31689	212.84526	14.58922
140	5.52243	1.63782	1.27977	773.13959	229.29431	15.14247
150	5.59118	1.63829	1.27996	838.67712	245.74344	15.67621
160	5.65551	1.63870	1.28012	904.88184	262.19257	16.19236
170	5.71595	1.63907	1.28026	971.71191	278.64175	16.69257
180	5.77295	1.63939	1.28039	1039.13074	295.09094	17.17821
190	5.82687	1.63968	1.28050	1107.10535	311.54010	17.65050
200	5.87803	1.63995	1.28060	1175.60645	327.98935	18.11048
210	5.92670	1.64018	1.28070	1244.60767	344.43863	18.55906
220	5.97311	1.64040	1.28078	1314.08533	360.88765	18.99705
230	6.01747	1.64060	1.28086	1384.01758	377.33713	19.42517
240	6.05994	1.64078	1.28093	1454.38501	393.78641	19.84405
250	6.10068	1.64094	1.28099	1525.16907	410.23563	20.25427
260	6.13982	1.64110	1.28105	1596.35315	426.68488	20.65635
270	6.17749	1.64124	1.28111	1667.92200	443.13412	21.05075
280	6.21379	1.64137	1.28116	1739.86145	459.58340	21.43790
290	6.24882	1.64149	1.28121	1812.15808	476.03268	21.81817
300	6.28267	1.64161	1.28125	1884.79956	492.48193	22.19193

To see this, re-write Equation (5-16) as

$$\sqrt{N}\ B(\alpha,N) = \frac{1}{\sqrt{N}} + \lambda\sqrt{\mathrm{var}[w_0(N)]} \tag{5-18}$$

In the infinite limit, the right side of this equation approaches $\lambda\pi/\sqrt{6}$. For $\gamma = 1$, we have $\alpha = \frac{1}{2}$. For this value of α, special relationships involving the Gamma functions in Equation (5-14) exist. In particular, from Equation (6.1.46) of Reference 3,

$$\lim_{N\to\infty} N^{1/2} \frac{\Gamma(N)}{\Gamma(N+1/2} = 1$$

We may thus write

$$\lim_{N\to\infty} \sqrt{N}\ B(1/2,N) = \Gamma(1/2) = \sqrt{\pi}$$

For $\lambda = \sqrt{6/\pi}$, the solution to Equation (5-18) as $N\to\infty$ is thus $\alpha = 1/2$. This value of λ is equivalent to a P_{FA} of .084. For larger λ (smaller and more realistic P_{FA}), a value of α smaller than 1/2 (γ greater than unity) will be required to satisfy (5-18), establishing our assertion.

As before, we now correct for the error introduced in the setting of the threshold. To do this, we determine the exact value of Λ at which the distribution of $v_0(N)$ is equal to $1 - P_{FA}$. For $N = 1$, the density of v is that of w, and Λ is readily found to be

$$\Lambda(N=1) = -\ln P_{FA} \tag{5-19}$$

For larger N, the density of $v_0(N)$ is the N-fold convolution of the density of $w_0(N)$. After convolving, this is then integrated to obtain the distribution function of $v_0(N)$.

The necessary calculations are most readily performed on a computer. The density of $w_0(N)$ is generated by sampling over a sufficiently large interval. For convenience, the mean value is removed, centering the density at zero. An FFT then transforms the density into the

characteristic function. Raising each term to the N^{th} power effects the convolution. An inverse FFT then results in the desired density. Numerical integration yields the distribution, whence the exact threshold (for given P_{FA}) can be determined.

From Equation (5-12), the variance, and hence standard deviation, of $v_0(N)$ is known. By normalizing v by this value, the distribution function just calculated is expressed in terms of a zero mean, unit variance random variable. By plotting this function on probability paper, one can determine the validity of the normal form used as an approximation. In Figure 5-2 is shown a true normal distribution (dashed line) and actual distributions of $v_0(N)$ for $N = 16$ and $N = 64$. As expected, in the tails, the error is noticeable. At .9999 ($P_{FA} = 10^{-4}$), for $N = 64$, the normal approximation is in error by roughly one-third of a standard deviation.

Having determined the exact threshold, $\Lambda(N)$, we equate to this the mean of $v_1(N)$. Solving for γ, we obtain the first correction to our initial approximation. This is shown as the dashed curve in Figure 5-1.

The second source of error arises from equating the mean to the median of $v_1(N)$. Using the same approach as for v_0, we may find the density of $v_1(N)$ as the N-fold convolution of the density of $w_1(N)$. The procedure is, however, not as straightforward since the density of $w_1(N)$ is a function of α(or, equivalently, of γ). A value of γ is chosen, the density of $v_1(N)$ is then calculated and integrated. The fiftieth percentile point is compared to the exact threshold $\Lambda(N)$. γ is varied until the two coincide. This value of signal-to-noise is the exact solution.

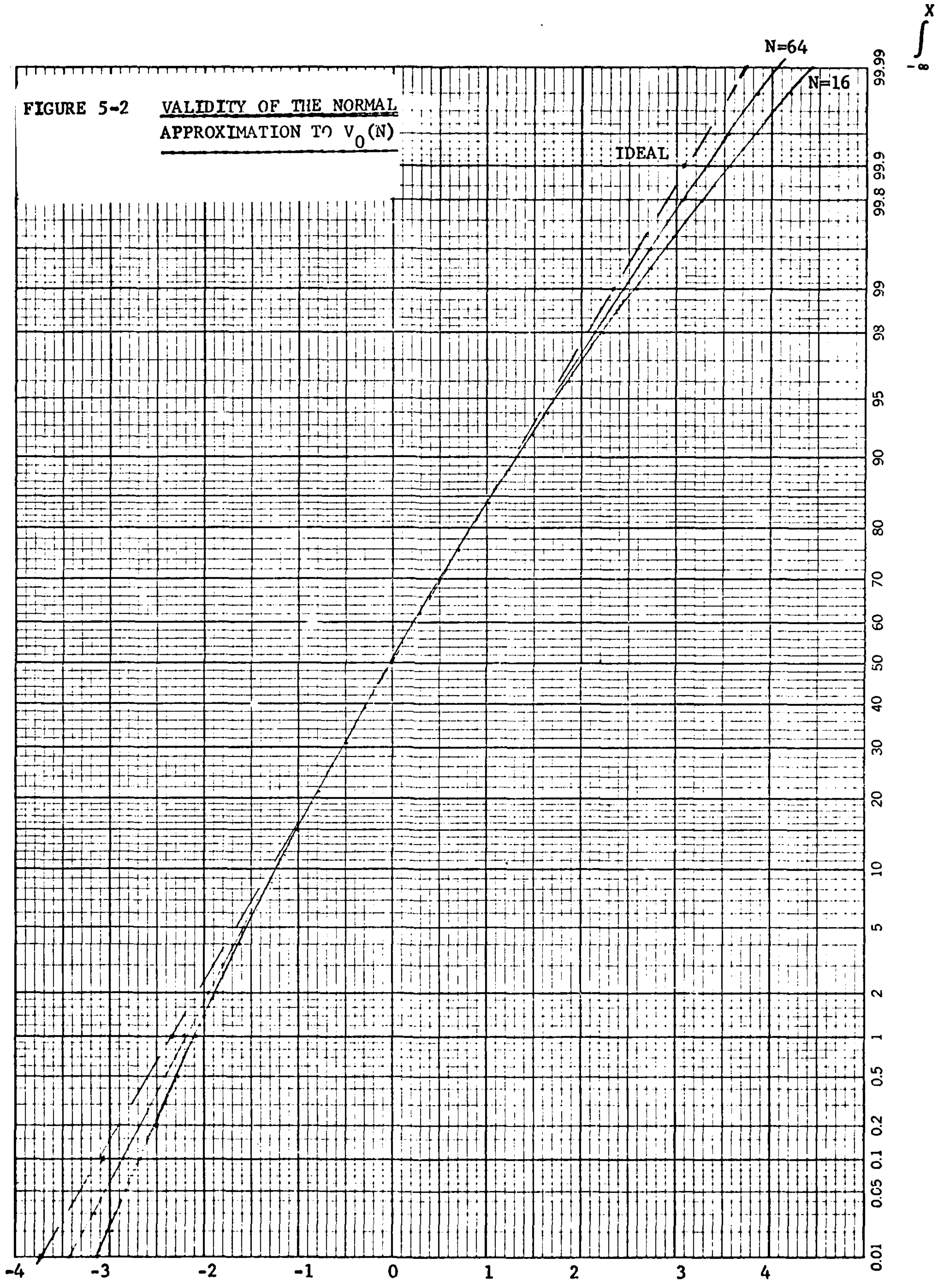

FIGURE 5-2
VALIDITY OF THE NORMAL APPROXIMATION TO $V_0(N)$
N=64
N=16
IDEAL
$\int_{-\infty}^{X}$
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
-4
-3
-2
-1
0
1
2
3
4
X IN STANDARD DEVIATIONS

For $N = 1$, the exact value of γ is already known to be 11.9 dB. For $N = 2$ and $N = 8$, exact values (of $\gamma = 9.1$ dB and $\gamma = 5.7$ dB) were found. As the corrections rapidly become small, further values were not found. The dashed and dotted curve in Figure 5-1, fitted through these points to the asymptotic result, thus represents the performance of the "greatest of" receiver.

SECTION 6

COMPARISONS AND CONCLUSIONS

As explained in Section 2.4, we are comparing an interceptor operating at a per-decision false-alarm probability of 10^{-5} against the intended receiver operating at 10^{-6}. The performance of the intended receiver operating at this P_{FA} is shown in Figure 4-4 of Reference 1. From this, we extract the values of SNR required by the intended receiver to have a fifty percent detection probability as a function of N. These are shown in Figure 6-1. Our comparison consists of simply subtracting the required SNR for the ED and GOF from that for the intended receiver. This yields the desired recognition differential, which is shown in Figure 6-2.

The negative values of recognition differential observed for small N reflect the difference in false alarm levels. If both the intended receiver and the interceptors were operating at the same false alarm level, then the recognition differential would be zero at $N = 1$ and strictly positive thereafter.

The ED is viewed as representing an upper bound on recognition differential, similarly, the GOF is a lower bound. Against any other form of intercept receiver, the recognition differential inherent in the chosen message structure will be bounded by these two curves. As the difference between the two is not very great, we conclude that there is no particular need to analyze in detail any of the other possible strategies available to an interceptor.

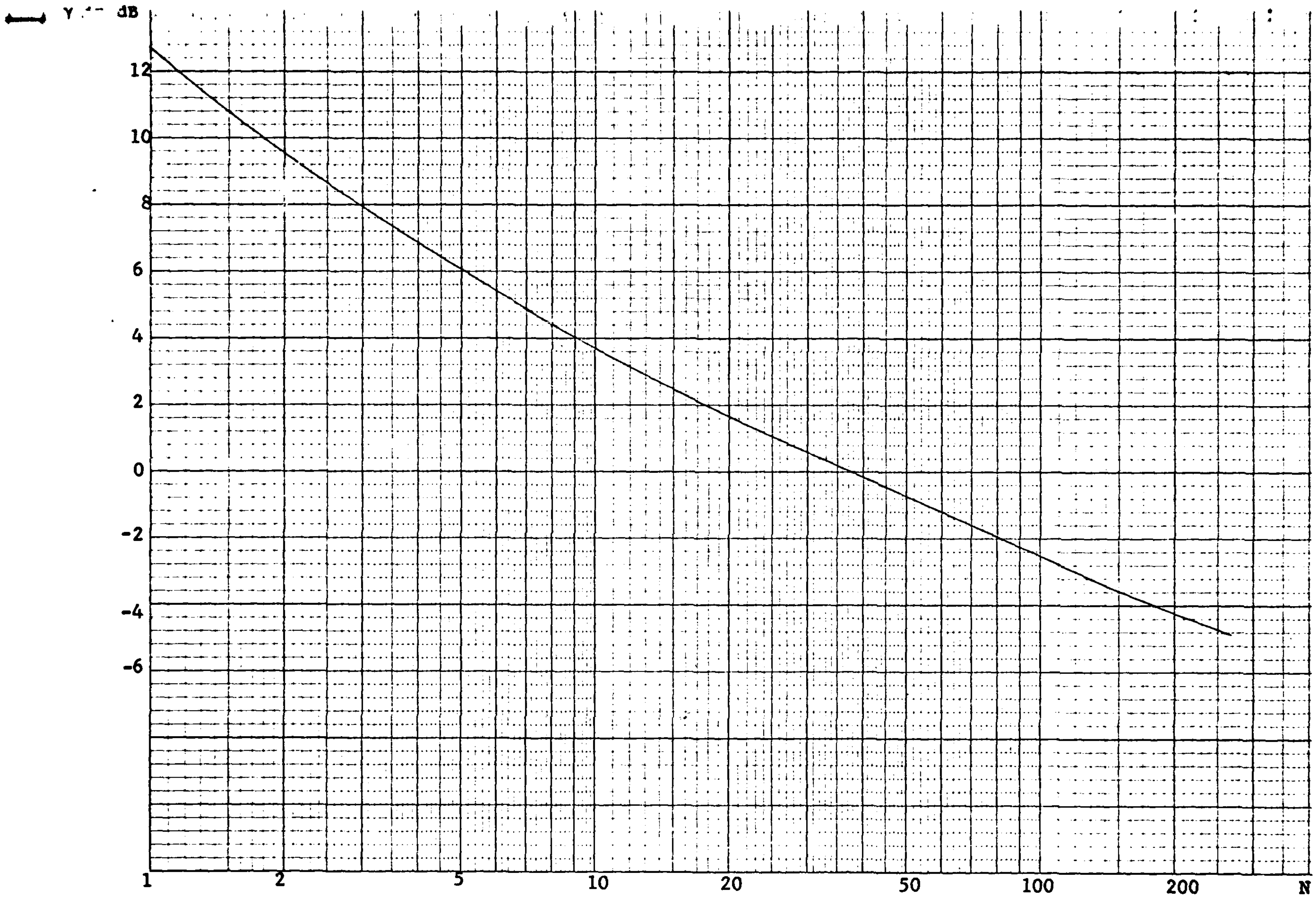

FIGURE 6-1 PERFORMANCE OF THE INTENDED RECEIVER: SNR FOR $P_D = .5$ @ $P_{FA} = 10^{-6}$

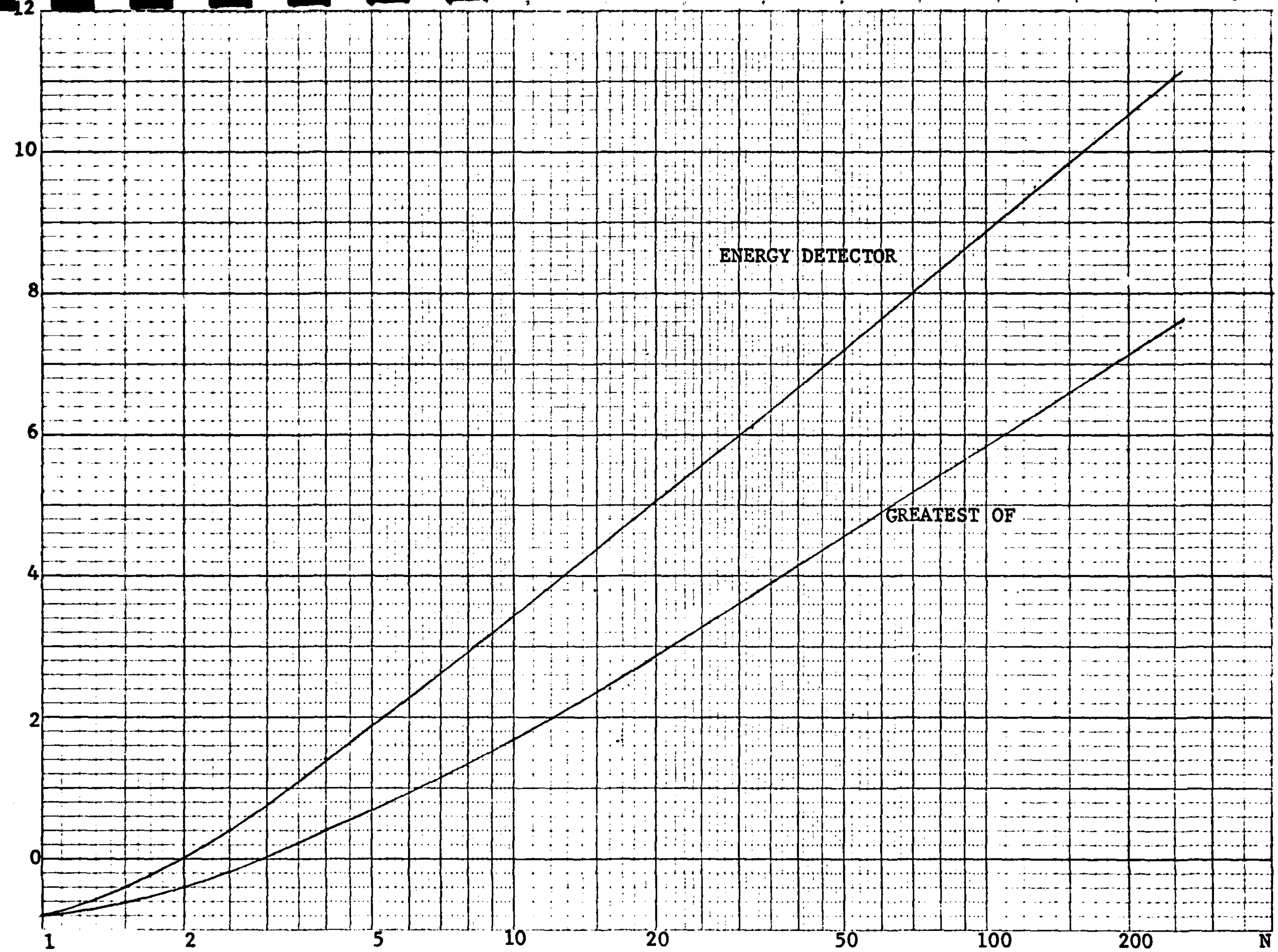

FIGURE 6-2 RECOGNITION DIFFERENTIAL

SECTION 7

REFERENCES

1. Pieper, J., "Detection of a Message Redundantly Transmitted Over a Fading Channel," Stein Associates Technical Note 76-01, 26 July 1976.

2. Pachares, James, "A Table of Bias Levels Useful in Radar Detection Problems," IRE Transactions on Information Theory, March 1958.

3. Abramowitz, M. and Stegun, I. A., "Handbook of Mathematical Functions," Dover Publications, Inc., New York, (1965).

4. Riordan, J., "Combinatorial Identities," John Wiley and Sons, Inc., (1968).

APPENDIX A

SOME SUMMATIONS INVOLVING BINOMIAL COEFFICIENTS

In this appendix we present the details behind the derivation of the moments of $w_0(N)$ and $w_1(N)$ for the "greatest of" receiver. First, we determine the expected value of w and w^2 under each hypothesis. We then simplify our results and determine expressions for the mean and variance. By definition,

$$E\left[w_0(N)\right] = \int_0^\infty w\, f_{0,N}(w)\, dw \qquad \text{(A-1)}$$

and

$$E\left[w_0^{\,2}(N)\right] = \int_0^\infty w^2\, f_{0,N}(w)\, dw \qquad \text{(A-2)}$$

From Equation (5-8), we have

$$f_{0,N}(w) = Ne^{-w}(1 - e^{-w})^{N-1} \qquad \text{(A-3)}$$

which we re-write as

$$f_{0,N}(w) = N\, e^{-w} \sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k (e^{-w})^k$$

by use of a binomial expansion. Utilizing a basic property of binomial coefficients, viz.

$$\frac{N}{k+1}\binom{N-1}{k} = \binom{N}{k+1} \qquad \text{(A-4)}$$

we have

$$f_{0,N}(w) = \sum_{k=0}^{N-1} \binom{N}{k+1} (-1)^k (k+1)(e^{-w})^{k+1}$$

which, via a change in index, becomes

$$f_{0,N}(w) = \sum_{k=1}^{N} \binom{N}{k} (-1)^{k-1} k (e^{-w})^k \qquad \text{(A-5)}$$

Substituting this into Equation (A-1) and interchanging the summation and integration, we have

$$E[w_0(N)] = \sum_{k=1}^{N} \binom{N}{k} (-1)^{k-1} k \int_0^{\infty} we^{-kw} dw$$

The integral is trivially shown to evaluate to $1/k^2$, hence,

$$E[w_0(N)] = \sum_{k=0}^{N} \binom{N}{k} (-1)^{k-1} \left(\frac{1}{k}\right) \quad \text{(A-6)}$$

The evaluation of $E[w_0{}^2(N)]$ is almost identical. The integral in this case is

$$\int_0^{\infty} w^2 e^{-kw} dw = \frac{2}{k^3}$$

so that

$$E[w_0{}^2(N)] = 2\sum_{k=1}^{N} \binom{N}{k} (-1)^{k-1} \left(\frac{1}{k}\right)^2 \quad \text{(A-7)}$$

We now consider H_1. Using Equation (5-10), we write

$$f_{1,N}(w) = f_{0,N-1}(w) - A + B$$

with $\quad A \triangleq (N-1)\ (1-e^{-w})^{N-2}\ e^{-(\alpha+1)w}$

and $\quad B \triangleq \alpha e^{-\alpha w}(1-e^{-w})^{N-1}$

Then,

$$E[w_1(N)] = E[w_0(N-1)] + \int_0^{\infty} w\ (B-A)dw \quad \text{(A-8)}$$

$$E[w_1{}^2(N)] = E[w_o{}^2(N-1)] + \int_0^{\infty} w^2\ (B-A)dw \quad \text{(A-9)}$$

We expand A and B.

$$B = \alpha \sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k (e^{-w})^k e^{-\alpha w}$$

$$A = (N-1)\sum_{k=0}^{N-2} \binom{N-2}{k} (-1)^k (e^{-w})^{k+1}\ e^{-\alpha w}$$

The expression for A is re-written employing the same technique used to change Equation (A-3) into (A-5).

$$A = \sum_{k=1}^{N-1} \binom{N-1}{k} (-1)^{k+1} k (e^{-w})^k \, e^{-\alpha w}$$

We note that for $k = 0$, the summand of A is zero, hence we may re-define the summation for A as commencing at $k = 0$. This allows us to combine A and B into a single summation:

$$B - A = \sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k \left[(k+\alpha) e^{-(k+\alpha)w}\right] \tag{A-10}$$

Substitution of Equation (A-10) into (A-8) and (A-9) followed by an interchange of the summation and integration yields results very similar in form to those stated in Equations (A-6) and (A-7).

$$E\left[w_1(N)\right] = E\left[w_0(N-1)\right] + \sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k \left(\frac{1}{k+\alpha}\right) \tag{A-11}$$

$$E\left[w_1^{\,2}(N)\right] = E\left[w_o^{\,2}(N-1)\right] + 2\sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k \left(\frac{1}{k+\alpha}\right)^2 \tag{A-12}$$

Note that the summations now start at $k = 0$, not $k = 1$. Thus, as $\alpha \to 0$, the leading term of the sums approaches ∞. This is to be expected as $\alpha = 0$ corresponds to an infinite signal-to-noise ratio, at which the expected value of the "greatest of" should be infinite.

We now simplify the summations for the mean and variance of $w_0(N)$ and the mean of $w_1(N)$.

The mean of $w_0(N)$ is defined by Equation (A-6). We use an inductive approach to express it more compactly. Let f_n represent the summation up to n, i.e.,

$$f_n \triangleq \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{1}{k} \tag{A-13}$$

By suitable manipulation, this is expressed in terms of f_{n-1}, whence the behavior as a function of n is observed. We use the readily proven recurrence relationship,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \tag{A-14}$$

which allows us to write

$$f_n = \sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k} \frac{1}{k} + \sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1} \frac{1}{k}$$

In the first summation, the $k = n$ term is zero since, by definition,

$$\binom{n-1}{n} = 0$$

Thus, the first summation may be written as from $k = 1$ to $n - 1$, which is directly seen to be f_{n-1}. Using Equation (A-4), the second sum is re-written as

$$\frac{1}{n} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k}$$

We then have

$$f_n = f_{n-1} + \frac{1}{n} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k}$$

However, since

$$[1 + (-1)]^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

we have the recurrence relationship

$$f_n = f_{n-1} + \frac{1}{n}$$

Noting that $f_1 = 1$, we finally obtain the closed form expression

$$f_n = \sum_{k=1}^{n} \frac{1}{k} \tag{A-15}$$

A similar procedure is now applied to the expected value of $w_0^2(N)$. Defining

$$g_n = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \left(\frac{1}{k}\right)^2$$

we readily establish that

$$g_n = g_{n-1} + \frac{1}{n} f_n \tag{A-16}$$

As $g_0 = 0$ and $g_1 = 1$, we have

$$g_n = \sum_{k=1}^{n} \frac{1}{k} f_k = \sum_{k=1}^{n} \sum_{m=1}^{k} \frac{1}{k} \frac{1}{m} \tag{A-17}$$

The variance of $w_0(N)$ is directly expressible in terms of f_N and g_N:

$$\mathrm{var}\left[w_0(N)\right] = 2g_N - f_N^2 \tag{A-18}$$

Expanding each of these as a double summation,

$$2g_N - f^2_N = 2\sum_{k=1}^{n} \sum_{m=1}^{k} \frac{1}{k} \frac{1}{m} - \sum_{k=1}^{N} \frac{1}{k} \sum_{m=1}^{N} \frac{1}{m}$$

$$= \sum_{k=1}^{N} \left[2\sum_{m=1}^{k} \frac{1}{km} - \sum_{m=1}^{N} \frac{1}{km}\right]$$

$$= \sum_{k=1}^{N} \left[\sum_{m=1}^{k} \frac{1}{km} - \sum_{m=k+1}^{N} \frac{1}{km}\right]$$

This is equivalent to

$$2g_N - f_N^2 = \sum_{k=1}^{N} \sum_{m=1}^{N} b_{k,m}$$

where
$$b_{k,m} = \begin{cases} +\frac{1}{km}, & m \leq k \\ -\frac{1}{km}, & m > k \end{cases}$$

Clearly, all the terms cancel except for those for which $k = m$. Thus,

$$\text{var}\left[w_0(N)\right] = \sum_{k=1}^{N} \left(\frac{1}{k}\right)^2 \tag{A-19}$$

Finally, we examine the mean of $w_1(N)$. We consider the summation in Equation (A-11) and claim that it is equivalent to a Beta function. This function is defined by

$$B(N,\alpha) = B(\alpha,N) \triangleq \int_0^1 (1-t)^{N-1} t^{\alpha-1} dt$$

Expanding the integrand,

$$B(N,\alpha) = \int_0^1 \left[\sum_{k=0}^{N-1} (-1)^k \binom{N-1}{k} t^k\right] t^{\alpha-1} dt$$

and interchanging the order of operation,

$$= \sum_{k=0}^{N-1} (-1)^k \binom{N-1}{k} \int_0^1 t^{k+\alpha-1} dt$$

or
$$B(N,\alpha) = \sum_{k=0}^{N-1} (-1)^k \binom{N-1}{k} \frac{1}{k+\alpha} \tag{A-20}$$

proving our claim. We may thus write

$$E\left[w_1(N)\right] = E\left[w_0(N-1)\right] + B(N,\alpha) \tag{A-21}$$

APPENDIX F

PROCESSOR SIZING

The processing required to implement the double-level soft decision decoder technique proposed for the HIDAR Receiver has been analyzed in order to provide a measure of assurance that the approach could be realized in hardware. The candidate processor selected is the Data Dependent Signal Processor (DDSP) recently developed by Stein Associates. This processor is soon to be announced as model AR-10/10 of the AR-10 family of modular high-speed digital processors. While the DDSP has certain special features that make the proposed processing more efficient and relatively easy to program, it is not unfair to say that the DDSP represents the general level of processing ability that is available with a typical modern signal processing computer.

Figure F-1 is a flowchart illustrating the main processing tasks that need to be performed to implement the double-level decoding. The flowchart is somewhat simplified, neglecting the facts that 1) double-level accumulation cannot begin until the start of message condition is detected, 2) first pass is simpler (there are no previous accumulations), and 3) the last pass is simpler since only one value of K (see below) is required. For timing analysis it is enough to recognize that as these special situations are all simpler, the maximum burdens occur during a typical pass as the message is being received.

The first task, accepting the input samples, may be made nearly transparent if special hardware is provided to put the incoming data in place in the working memories of the DDSP using a background DMA data

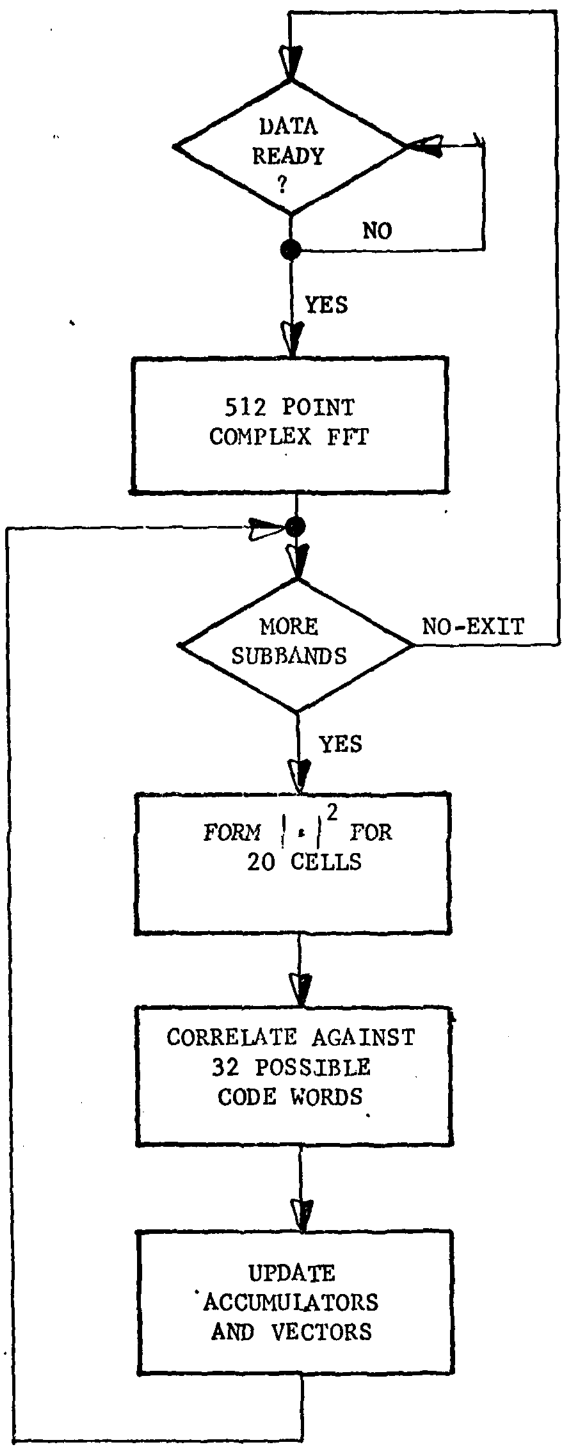

FIGURE F-1 (U) BASIC DECODER FLOW (U)

transfer. Since this technique is unique to the DDSP, we will take a worse-case assumption that such hardware is not present requiring the program to sense that the data is ready on a sample-by-sample basis. Once the processing of a block of data is started, it will be necessary to interrupt processing at regular intervals to save input data and store it away in a software FIFO (first-in, first-out) memory for later use. When the processing for the previous block is complete, the program must first empty the accumulated samples in the FIFO before sensing new samples at the input. Such a routine, coded for the DDSP as part of a previously developed program, times out at approximately 1.5 us per sample. Since there are assumed to be forty tones per subband, the number of samples per pass is 40x7=280 requiring 280 x 1.5 us $\approx$ .4 ms.

Once the data is taken, a complex FFT must be performed using a number of points equal to the incoming sample set rounded up to the next binary increment. The FFT has been coded for the DDSP, requiring 12 instructions (1.62 ms) for the butterfly calculation that is the heart of the process. With approximately 10% additional processing to set up the parameters and indexes for each stage, the worst case (512 points) total time required is (1.62x2304)1.1$\approx$4.1 ms.

After the FFT, the program operates on a subband basis, calculating the $(\text{magnitude})^2$ for each frequency (using only odd or even), correlating again all possible codes and updating the double-level accumulators and vectors.

Of these processes, forming the square of the magnitude is most straightforward. This process requires two multiplications (two instruction

times each) plus a summation with one or two more instructions to fetch and restore the data. By overlaying the processing for one frequency with the next, the total processing can be reduced to five instructions per sample for a total burden of 5 instructions x 20 frequencies plus 20 instructions to initialize the loop. The total burden is 120 x 135 ns x 7 $\approx$ .1 ms.

The correlation process is somewhat more complex since it consists of selected accumulations of ten of the twenty input magnitudes where 32 different patterns for selection are used. The innermost part of this program consists of a subroutine that reads in the current pattern word, examines the next bit and, if it is ONE, adds a corresponding magnitude into an accumulator. This routine is estimated to require 4 instructions if the pattern bit is ZERO or 6 instructions if the pattern bit is ONE. Since half the bits are ONE and half ZERO, the average is five instructions. With twenty pattern bits per code, 32 codes, five instructions per code, 135 ns per instruction and 7 subbands, the total, allotting 10 percent for overhead, is (20 x 32 x 5 x 135ns x 7)1.1 $\approx$ 3.3 ms.

Most complex of all is the double-level decoding. The innermost loop of this process requires the addition of the ith accumulator with the jth correlation sum where the sum of i and j is K(modulo 32). After the 32 summations that meet this criteria are formed the largest is selected and the new sum replaces the old and the number j is added to the decision vector. This process is repeated for all 32 possible values of K.

The program to perform this inner loop must first calculate the j associated with each i, fetch the corresponding accumulator and correlation sum, add them, test to see if it is larger than previous

sums and, if so, save the sum, i and j. This routine has been tentatively programmed, requiring 12 instructions. Since it is in a double loop each repeated 32 times, the total requirement is $\{(12 \text{ instructions} \times 32)1.1 \times 32 \times 135 \text{ ns/instruction} \times 7 \text{ subbands}\} 1.1 \approx 14.1$ ms.

Since the total burden is the sum of the individual requirements it is estimated that the total processing is approximately .4ms + 4.1 ms + .1 ms + 3.3 ms + 14.1 ms = 22.0 ms. This processing is repeated every 30 ms. Thus, the burden may also be expressed as 22/30 ≈ 73% of available processing time, a figure that provides an acceptable level of margin allowing for the growth and additional tasks that usually must be added as a system moves from the analysis stages to detailed implementation.